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Method of spectral mappings in the inverse problem theory

Sturm-Liouville operators on a finite interval. Consider the BVP L:

$$\ell y := -y'' + q(x)y = \lambda y,\ 0 < x < \pi, \quad q(x) \in L(0,\pi), \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0.$$

Let $\varphi(x,\lambda), S(x,\lambda)$ be the solutions of Eq. (1) with the conditions $\varphi(0,\lambda) = 1,\ \varphi'(0,\lambda) = h,\ S(0,\lambda) = 0,\ S'(0,\lambda) = 1$. Denote

$$\Delta(\lambda) := V(\varphi), \quad \{\lambda_n\}_{n\ge 0}, \quad \alpha_n = \int_0^\pi \varphi^2(x,\lambda_n)\,dx.$$

IP 1. (V. Marchenko, 1950). Given $\{\lambda_n, \alpha_n\}$, construct q, h, H.
Consider the BVP L_1 for (1) with the conditions $y(0) = V(y) = 0$.

$$\delta(\lambda) := V(S), \quad \{\nu_n\}_{n\ge 0}.$$

IP 2. (G. Borg, 1946). Given $\{\lambda_n, \nu_n\}$, construct q, h, H.

3) **The Weyl function.** Let $\Phi(x,\lambda)$ be the solution of Eq.(1) with the conditions $U(\Phi)=1,\ V(\Phi)=0$. We set $M(\lambda):=\Phi(0,\lambda)$.

IP 3. Given $M(\lambda)$, construct q, h, H.

$$M(\lambda) = -\frac{\delta(\lambda)}{\Delta(\lambda)},\ M(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n(\lambda-\lambda_n)}, \tag{2}$$

$$\Delta(\lambda) = \pi(\lambda_0-\lambda)\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{n^2}, \quad \delta(\lambda) = \prod_{n=0}^{\infty}\frac{\nu_n-\lambda}{(n+1/2)^2}. \tag{3}$$

IP 3 is equivalent to IP 1 and IP 2.

Sturm-Liouville operators on the half-line. Consider the BVP L:

$$\ell y := -y'' + q(x)y = \lambda y,\ x > 0, \quad q(x) \in L(0,\infty), \tag{4}$$

$$U(y) := y'(0) - hy(0) = 0.$$

Let $\Phi(x,\lambda)$ be the solution of (4) under the conditions

$$U(\Phi) = 1, \quad \Phi(x,\lambda) = O(\exp(i\rho x)),\ x \to \infty,$$

where $\lambda = \rho^2,\ Im\,\rho \geq 0$. Denote $M(\lambda) := \Phi(0,\lambda)$.

IP 4. Given $M(\lambda)$, construct $q(x)$ and h.

$$M(\lambda) = \int_{-\infty}^{\infty} \frac{d\sigma(\mu)}{\lambda - \mu}. \tag{5}$$

Transformation operator method: V.Marchenko, B.Levitan, 1950-51.

Let $\lambda = \rho^2$. The following representation is valid

$$\varphi(x, \lambda) = \cos \rho x + \int_0^x G(x, t) \cos \rho t \, dt, \tag{6}$$

$$q(x) = \frac{d}{dx} G(x, x), \quad h = G(0, 0). \tag{7}$$

Case 1: finite interval. Take a model BVP $\tilde{L}$ with $\tilde{q} = 0$, $\tilde{h} = \tilde{H} = 0$. Then $\tilde{\lambda}_n = n^2$, $n \geq 0$. Consider the function

$$F(x, t) = \sum_{n=0}^{\infty} \left(\frac{\cos \rho_n x \cos \rho_n t}{\alpha_n} - \frac{\cos nx \cos nt}{\tilde{\alpha}_n} \right)$$

$$= \frac{1}{2\pi i} \int_\gamma \cos \rho x \cos \rho t \hat{M}(\lambda) \, d\lambda, \quad \hat{M} := M - \tilde{M}, \tag{8}$$

$\tilde{\alpha}_n = \pi/2$ $(n > 0)$; $\tilde{\alpha}_0 = \pi$; γ is a contour encircling the spectra of L and $\tilde{L}$.

Theorem 2. *For each fixed x, the kernel $G(x,t)$ from representation (6) satisfies the linear integral equation*

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)\,ds = 0,\ 0 < t < x. \qquad (9)$$

Case 2: the half-line. Take a model BVP $\tilde{L}$ with $\tilde{q}(x) = 0,\ \tilde{h} = 0.$ Consider the function

$$F(x,t) = \frac{1}{2\pi i}\int_\gamma \cos\rho x \cos\rho t \hat{M}(\lambda)\,d\lambda, \qquad (10)$$

where γ is a contour encircling the spectra of L and $\tilde{L}$.
Theorem 2 remains true with (10) instead of (8).

In both cases $q(x)$ and h can be constructed by (7).

Method of Spectral Mappings: V. Yurko, 1985-1986

[1] Yurko V.A. Recovery of nonselfadjoint differential operators on the half-line from the Weyl matrix. Matem. Sbornik, vol.182, no.3 (1991), 431-456 (Math. USSR Sbornik, vol.72, no.2 (1992), 413-438).

[2] Yurko V.A., Inverse Spectral Problems for Differential Operators and their Applications, Gordon and Breach, New York, 1998.

[3] Freiling G. and Yurko V.A., Inverse Sturm-Liouville Problems and their Applications, NOVA Science Publishers, New York, 2001.

[4] Yurko V.A., Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.

[5] Yurko V.A., Introduction to the theory of inverse spectral problems. Moscow, Fizmatlit, 2007, 384pp.

Higher-order differential equations:

$$\ell y := y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)} = \lambda y, \; n > 2. \tag{11}$$

Method of Spectral Mappings for Sturm-Liouville operators

Let the Weyl function $M(\lambda)$ be given. Choose a model BVP $\tilde{L}$ with $\tilde{q}$ and $\tilde{h}$ (for example, one can take $\tilde{q}(x) = 0,\ \tilde{h} = 0\ \to\ \tilde{\varphi}(x, \lambda) = \cos \rho x$). Denote

$$\tilde{r}(x, \lambda, \mu) = \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{M}(\mu) = \int_0^x \tilde{\varphi}(t, \lambda) \tilde{\varphi}(t, \mu)\, dt\, \hat{M}(\mu),$$

where $\hat{M} := M - \tilde{M},\ \langle y, z \rangle := yz' - y'z$.

Theorem 3. *The following relation holds*

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \frac{1}{2\pi i} \int_\gamma \tilde{r}(x, \lambda, \mu) \varphi(x, \mu)\, d\mu. \qquad (12)$$

Here γ is a contour encircling the spectra.

Consider the Banach space $C(\gamma)$ of continuous bounded functions $z(\lambda),\ \lambda \in \gamma$, with the norm $\|z\| = \sup_{\lambda\in\gamma} |z(\lambda)|$.

Theorem 4. *For each x, Eq. (12) has a unique solution $\varphi(x,\lambda) \in C(\gamma)$.*

Theorem 5. *The following relations hold*

$$q(x) = \tilde{q}(x) - 2\varepsilon_0'(x),\ h = \tilde{h} - \varepsilon_0(0), \tag{13}$$

$$\varepsilon_0(x) := \frac{1}{2\pi i} \int_\gamma \tilde{\varphi}(x,\mu)\varphi(x,\mu)\hat{M}(\mu)\, d\mu.$$

Algorithm 1. *Let the function $M(\lambda)$ be given.*
(1) Choose $\tilde{L}$ and construct $\tilde{\varphi}$ and $\tilde{r}$.
(2) Find $\varphi(x,\lambda)$ by solving equation (12).
(3) Construct $q(x)$ and h via (13).

Proof of Theorem 3. Consider the functions

$$P_{11} = \varphi\tilde{\Phi}' - \Phi\tilde{\varphi}', \quad P_{12} = \Phi\tilde{\varphi} - \varphi\tilde{\Phi}. \tag{14}$$

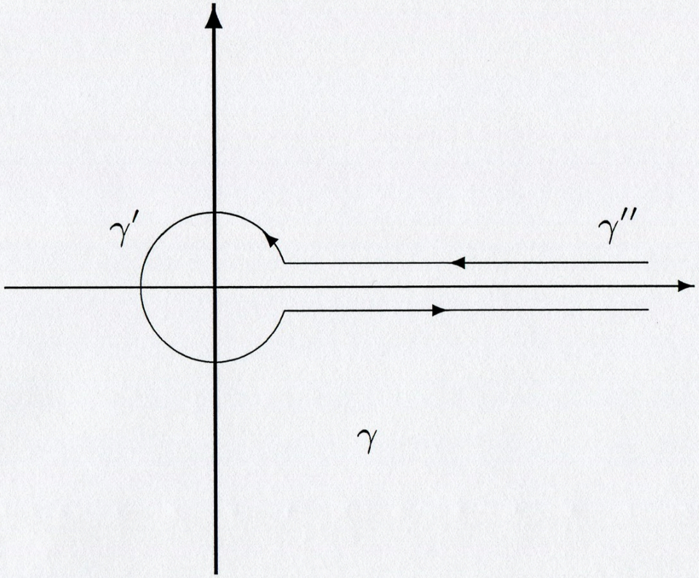
Since $\varphi\Phi' - \Phi\varphi' = 1$, it follows from (14) that

$$\varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda).$$

One has

$$P_{1k}(x,\lambda) - \delta_{1k} = O(\rho^{-1}),\ |\lambda| \to \infty,\ \lambda = \rho^2, \tag{15}$$

where δ_{jk} is the Kronecker symbol. For definiteness we consider the case of the half-line. For a finite interval the arguments are similar.



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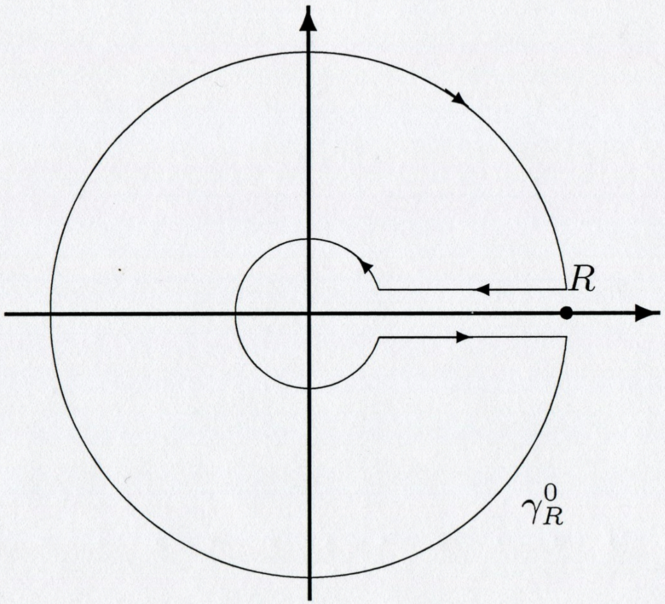


R
γ_R^0

Denote by Λ the discrete spectrum; it is a bounded set. In the λ - plane we consider the contour $\gamma = \gamma' \cup \gamma''$ (with counterclockwise circuit), where γ' is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$, and γ'' is the two-sided cut along the arc $\{\lambda : \ \lambda > 0, \ \lambda \notin int\, \gamma'\}$. Denote $J_\gamma = \{\lambda : \ \lambda \notin \gamma \cup int\, \gamma'\}$. Consider the contour $\gamma_R = \gamma \cap \{\lambda : \ |\lambda| \le R\}$ with counterclockwise circuit, and also consider the contour $\gamma_R^0 = \gamma_R \cup \{\lambda : \ |\lambda| = R\}$ with clockwise circuit. By Cauchy's integral formula,

$$P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_R^0} \frac{P_{1k}(x,\mu) - \delta_{1k}}{\lambda - \mu}\, d\mu,$$

where $\lambda \in int\, \gamma_R^0$. Using (15) we get

$$\lim_{R\to\infty} \int_{|\mu|=R} \frac{P_{1k}(x,\mu) - \delta_{1k}}{\lambda - \mu}\, d\mu = 0 \quad \rightarrow$$

$$P_{1k}(x,\lambda) = \delta_{1k} + \frac{1}{2\pi i}\int_\gamma \frac{P_{1k}(x,\mu)}{\lambda-\mu}\,d\mu,\ \ \lambda \in J_\gamma.$$

Since $\varphi = P_{11}\tilde{\varphi} + P_{12}\tilde{\varphi}'$, one has

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{\tilde{\varphi}(x,\lambda)P_{11}(x,\mu) + \tilde{\varphi}'(x,\lambda)P_{12}(x,\mu)}{\lambda-\mu}\,d\mu.$$

Taking (14) into account we get

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma (\tilde{\varphi}(x,\lambda)(\varphi(x,\mu)\tilde{\Phi}'(x,\mu) - \Phi(x,\mu)\tilde{\varphi}'(x,\mu)) +$$

$$\tilde{\varphi}'(x,\lambda)(\Phi(x,\mu)\tilde{\varphi}(x,\mu) - \varphi(x,\mu)\tilde{\Phi}(x,\mu))\frac{d\mu}{\lambda-\mu}.$$

Using the relations $\Phi = S + M\varphi,\ \tilde{\Phi} = \tilde{S} + \tilde{M}\tilde{\varphi}$, we arrive at (12), since the terms with $S(x,\mu)$ vanish by Cauchy's theorem.

Higher-order operators, 1985-1986. Consider the equation

$$\ell y := y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)} = \lambda y, \; n > 2. \qquad (1)$$

Let $\lambda = \rho^n$. The ρ- plane can be partitioned into sectors S_ν of angle $\frac{\pi}{n}$ $\left(S_\nu := \{\rho : \arg\rho \in \left(\frac{\nu\pi}{n}, \frac{(\nu+1)\pi}{n}\right)\}, \; \nu = \overline{0, 2n-1}\right)$ in each of which the roots $R_1, R_2, \ldots, R_n$ of the equation $R^n - 1 = 0$ can be numbered in such a way that

$$Re(\rho R_1) < \ldots < Re(\rho R_n), \quad \rho \in S_\nu.$$

Let $\Phi_m(x, \lambda)$, $m = \overline{1, n}$, be the solutions of Eq. (1) with the conditions

$$\Phi_m^{(n-\xi)}(0, \lambda) = \delta_{\xi m}, \quad \xi = \overline{1, m},$$

$$\Phi_m(x, \lambda) = O(\exp(\rho R_m x)), \quad x \to \infty, \quad \rho \in S_\nu.$$

Denote $M(\lambda) = [M_{mk}(\lambda)]_{m,k=\overline{1,n}}$, $M_{mk}(\lambda) = \Phi_m^{(n-k)}(0,\lambda)$. The matrix $M(\lambda)$ is called the Weyl matrix for ℓ.

Example: n=4.

$\Phi_1'''(0,\lambda) = 1,\ \Phi_1(x,\lambda) = O(\exp(\rho R_1 x)),$

$\Phi_2'''(0,\lambda) = 0,\ \Phi_2''(0,\lambda) = 1,\ \Phi_2(x,\lambda) = O(\exp(\rho R_2 x)),$

$\Phi_3'''(0,\lambda) = \Phi_3''(0,\lambda) = 0,\ \Phi_3'(0,\lambda) = 1,\ \Phi_3(x,\lambda) = O(\exp(\rho R_3 x)),$

$\Phi_4'''(0,\lambda) = \Phi_4''(0,\lambda) = \Phi_4'(0,\lambda) = 0,\ \Phi_4(0,\lambda) = 1,$

$$M(\lambda) = \begin{bmatrix} 1 & M_{12}(\lambda) & M_{13}(\lambda) & M_{14}(\lambda) \\ 0 & 1 & M_{23}(\lambda) & M_{24}(\lambda) \\ 0 & 0 & 1 & M_{34}(\lambda) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Inverse problem. Given the Weyl matrix $M(\lambda)$, construct ℓ.

Let $\Gamma_\pm := \{\lambda : \pm\lambda \geq 0\}$, and $\Pi_\pm$ be the λ- plane with a cut along $\Gamma_\pm$.

Theorem 1. *The Weyl matrix $M(\lambda)$ has the following properties:*
1) $M_{mk}(\lambda) = \delta_{mk},\ m \geq k$.
2) The functions $M_{mk}(\lambda)$ are analytic in $\Pi_{(-1)^{n-m}}$ with the exception of at most countable bounded sets Λ'_{mk} of poles and are continuous in $\bar{\Pi}_{(-1)^{n-m}}$ with the exception of bounded sets Λ_{mk}.
3) $M_{mk}(\lambda) = O(\rho^{m-k})$ as $|\lambda| \to \infty$.
4) The functions $(M_{mk} - M_{m,m+1}M_{m+1,k})(\lambda)$ are analytic for $\lambda \in \Gamma_{(-1)^{n-m}} \setminus \Lambda$, where $\Lambda = \bigcup_{m,k} \Lambda_{mk}$.

Let $M(\lambda)$ be the Weyl matrix for ℓ. Take a model operator $\tilde{\ell}$. In the λ-plane we consider the contour $\gamma = \gamma_{-1} \cup \gamma_0 \cup \gamma_1$ (with a counterclockwise circuit), where γ_0 is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$ (i.e. $\Lambda \cup \tilde{\Lambda} \cup \{0\} \subset \mathrm{int}\gamma_0$), and $\gamma_{\pm 1}$ is a two-sided cut along the ray $\{\lambda : \ \pm\lambda > 0, \ \lambda \notin \mathrm{int}\gamma_0\}$. Denote

$$\varphi(x,\lambda) = [\chi((-1)^{n-k+1}\lambda)\Phi_k(x,\lambda)]_{k=\overline{2,n}},$$

where $\chi_{\pm 1}(\lambda) = 1$ for $\lambda \in \gamma_0 \cup \gamma_{\pm 1}$, $\chi_{\pm 1}(\lambda) = 0$ for $\lambda \in \gamma_{\mp 1}$.

Theorem 2. *For each fixed $x \geq 0$, the vector $\varphi(x,\lambda)$ is a solution of the linear singular integral equation*

$$\tilde{\varphi}(x,\lambda) = Q(\lambda)\varphi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \frac{H(x,\lambda,\mu)}{\mu - \lambda}\,\varphi(x,\mu)\,d\mu, \ \lambda \in \gamma, \qquad (2)$$

where $Q(\lambda)$ and $H(x,\lambda,\mu)$ are constructed from $\tilde{\ell}$ and $M(\lambda)$.

Denote $\Omega(x,\lambda) = \operatorname{diag}\left[\rho^{n-k}\exp(-\rho R_k x)\right]_{k=\overline{2,n}}$,
$\gamma'' = \{\lambda:\ \lambda \in \gamma_1 \cup \gamma_{-1},\ d(\lambda,\gamma_0) \geq \alpha_0 > 0\}$, $\gamma' = \gamma \setminus \gamma''$, where
$d(\lambda,\gamma_0) := \inf |\lambda - \mu|,\ \mu \in \gamma_0$. We introduce the Banach space
$B = L_2^{n-1}(\gamma') \oplus L_\infty^{n-1}(\gamma'')$ of vector-valued functions $z(\lambda) = [z_j(\lambda)]_{j=\overline{1,n-1}}$,
$\lambda \in \gamma$ with the norm

$$\|z\|_B = \sum_{j=1}^{n-1} \Big(\|z_j\|_{L_2(\gamma')} + \|z_j\|_{L_\infty(\gamma'')} \Big).$$

Theorem 3. *For each $x \geq 0$, equation (2) has a unique solution in the class $\Omega(x,\lambda)\varphi(x,\lambda) \in B$, and $\sup\limits_x \|\Omega(x,\lambda)\varphi(x,\lambda)\|_B < \infty$.*

Algorithm. 1) Choose a model operator $\tilde{\ell}$.
2) Construct the matrices $H(x,\lambda,\mu), Q(\lambda), \tilde{\varphi}(x,\lambda)$, $x \geq 0,\ \lambda,\mu \in \gamma$.
3) Find $\varphi(x,\lambda)$, $x \geq 0$, $\lambda \in \gamma$ by solving the main equation (2).
4) Construct ℓ.

Inverse problems for systems: V. Yurko, 2004-2005.

Consider the system

$$\ell Y(x) := Q_0 Y'(x) + Q(x)Y(x) = \rho Y(x), \quad x > 0, \tag{1}$$

$Q_0 = \mathrm{diag}[q_k]_{k=\overline{1,n}}$, $Q(x) = [q_{kj}(x)]_{k,j=\overline{1,n}}$, $q_{kk}(x) \equiv 0$. Let $\beta_k = 1/q_k$. The ρ- plane can be partitioned into sectors $S_j = \{\rho :\ \arg\rho \in (\theta_j, \theta_{j+1})\}$, $j = \overline{0, 2r-1}$, $0 \le \theta_0 < \theta_1 < \ldots < \theta_{2r-1} < 2\pi$, in each of which there exists a permutation $i_k = i_k(S_j)$ of the numbers $1, \ldots, n$, such that for the numbers $R_k = R_k(S_j)$ of the form $R_k = \beta_{i_k}$ one has

$$\mathrm{Re}(\rho R_1) < \ldots < \mathrm{Re}(\rho R_n), \quad \rho \in S_j. \tag{2}$$

Let the matrix $h = [h_{\xi\nu}]_{\xi,\nu=\overline{1,n}}$, $\det h \neq 0$ be given. We introduce the linear forms $U(Y) = [U_\xi(Y)]^T_{\xi=\overline{1,n}}$ by $U(Y) = hY(0)$.

Denote $\Omega^0_{mk}(j_1, \dots, j_m) = \det[h_{\xi, j_\nu}]_{\xi=\overline{1,m-1},k;\ \nu=\overline{1,m}}$. Let

$$\Omega^0_{mm}(i_1, \dots, i_m) \neq 0,\ m = \overline{1, n-1},\ j = \overline{0, 2r-1}.$$

This condition is called the information condition for $L = (\ell, U)$.

Let $\Phi_m(x, \rho) = [\Phi_{km}(x, \rho)]^T_{k=\overline{1,n}}$, $m = \overline{1, n}$, be solutions of system (1) under the conditions

$$U_\xi(\Phi_m) = \delta_{\xi m},\ \xi = \overline{1, m},$$

$$\Phi_m(x, \rho) = O(\exp(\rho R_m x)),\ x \to \infty,\ \rho \in S_j.$$

Let $M_{m\xi}(\rho) = U_\xi(\Phi_m)$, $M(\rho) = [M_{m\xi}(\rho)]_{m,\xi=\overline{1,n}}$.

Inverse problem. Given $M(\rho)$, construct Q and h.

Theorem 1. *The specification of the Weyl matrix $M(\rho)$ uniquely determines the potential $Q(x)$ and the matrix h.*

Denote $\Gamma_j = \{\rho : \arg\, \rho = \theta_j\}$, $j = \overline{0, 2r-1}$, $\Gamma_{2r} := \Gamma_0$, $\Sigma = \bigcup\limits_{j=0}^{2r-1} S_j$ - the ρ-plane without the cuts along the rays Γ_j. Let $\Gamma_j^{\pm} = \{\rho :\ \arg \rho = \theta_j \pm 0\}$ be the sides of the cuts.

Fix $j = \overline{0, 2r-1}$. For $\rho \in \Gamma_j$, strict inequalities from (2) in some places become equalities. Let $m_i = m_i(j)$, $p_i = p_i(j)$, $i = \overline{1, s}$ be such that for $\rho \in \Gamma_j$: $\mathrm{Re}(\rho R_{m_i - 1}) < \mathrm{Re}(\rho R_{m_i}) = \ldots = \mathrm{Re}(\rho R_{m_i + p_i}) < \mathrm{Re}(\rho R_{m_i + p_i + 1})$, $R_k = R_k(S_j)$. Let $N_j := \{m : m = \overline{m_1, m_1 + p_1 - 1}, \ldots, \overline{m_s, m_s + p_s - 1}\}$, $J_m := \{j :\ m \in N_j\}$, $\gamma_m = \bigcup\limits_{j \in J_m} \Gamma_j$, and let $\Sigma_m = \mathbf{C} \setminus \gamma_m$ be the ρ- plane without the cuts along the rays from γ_m. Clearly, the domain $\Sigma_m = \bigcup\limits_{\nu} S_{m\nu}$ consists of the sectors $S_{m\nu}$, each of which is a union of several sectors S_j with the same set $\{R_\xi\}_{\xi = \overline{1, m}}$.

We introduce the functions $B^{\xi}_{mk}(\rho)$ by
$B^0_{mk}(\rho) = M_{mk}(\rho),\ B^{\xi}_{mk}(\rho) = B^{\xi-1}_{mk}(\rho) - B^{\xi-1}_{m,m+\xi}(\rho) B^0_{m+\xi,k}(\rho),$
$\xi = \overline{0, n-2},\ m = \overline{1, n-\xi-1},\ k = \overline{m+\xi+1, n}.$

Denote by $\mathcal{M}$ the set of functions $M(\rho) = [M_{mk}(\rho)]_{m,k=\overline{1,n}}$ such that:
1) $M_{mk}(\rho) \equiv \delta_{mk}$ for $m \geq k$;
2) The function $M_{mk}(\rho), k > m$, is analytic in Σ_m with the exception of an at most countable bounded set Λ'_m of poles, and are continuous in $\overline{\Sigma}_m$ with the exception of a bounded set Λ_m;
3) The function $B^{m-\nu}_{\nu k}(\rho)$ is analytic on
$\Gamma_j \setminus \Lambda'_m, j \notin J_m, 1 \leq \nu \leq m \leq n-1, m+1 \leq k \leq n$;
4) $M_{mk}(\rho) = \mu^0_{mk}(S_j) + O(\rho^{-1}),\ \mu^0_{mk}(S_j) = \dfrac{\Omega^0_{mk}(i_1, \dots, i_m)}{\Omega^0_{mm}(i_1, \dots, i_m)},\ \rho \in \overline{S}_j.$

Theorem 2. *If $M(\rho)$ is the Weyl matrix for a pair $L = (\ell, U)$, then $M(\rho) \in \mathcal{M}$.*

Denote $\Lambda := \Lambda_1 \cup \ldots \cup \Lambda_{n-1}$. Let $M(\rho)$ be the Weyl matrix for $L = (\ell, U)$. We choose a pair $\tilde{L} = (\tilde{\ell}, \tilde{U})$ such that $M(\rho) - \tilde{M}(\rho) = O(\rho^{-1})$, $|\rho| \to \infty$. In the ρ- plane we consider the contour $\omega^* := \omega^0 \cup \omega^1$, where ω^0 is a bounded closed contour encircling the set $\Lambda \cup \tilde{\Lambda} \cup \{0\}$ (i.e. $\Lambda \cup \tilde{\Lambda} \cup \{0\} \subset \operatorname{int} \omega^0$), and $\omega^1 = \bigcup_{j=0}^{2r-1} \omega_j^1$, $\omega_j^1 := \{\rho : \ \rho \in \Gamma_j \setminus \omega^0\}$. Denote

$$\varphi(x, \rho) = \begin{cases} [\Phi^+(x, \rho), \Phi^-(x, \rho)], & \rho \in \omega^1, \\ \Phi(x, \rho), & \rho \in \omega^0, \end{cases}$$

where $\Phi^\pm := \Phi_{|\omega^\pm}$, $\omega^\pm = \bigcup_{j=0}^{2r-1} \omega_j^\pm$, $\omega_j^\pm = \Gamma_j^\pm \setminus \operatorname{int} \omega^0$.

We consider the Banach space $\mathcal{B}_p := \{f(\rho) : \ f(\rho)\rho^{-1} \in L_p(\omega^*)\}$, $p > 1$ with the norm $\|f\|_{\mathcal{B}_p} := \|f(\rho)\rho^{-1}\|_{L_p(\omega^*)}$.

Theorem 3. *Let $M(\rho)$ be the Weyl matrix for the pair $L = (\ell, U)$. The following relation is valid for $\rho \in \omega^*$:*

$$\tilde{\varphi}(x, \rho) = \varphi(x, \rho) S(\rho) + \frac{1}{2\pi i} \int_{\omega^*} \varphi(x, \mu) r(x, \mu, \rho) \, d\mu, \qquad (3)$$

where $S(\rho)$ and $r(x, \rho, \mu)$ are constructed from $\tilde{\ell}$ and $M(\rho)$.

For each fixed $x \geq 0$, equation (3) has the unique solution $\varphi(x, \rho)$ in the class $\varphi(x, \rho) D(x, \rho) \in \mathcal{B}_p$ for each $p > 1$; and $\sup_{x \geq 0} \|\varphi(x, \rho) D(x, \rho)\|_{\mathcal{B}_p} < \infty$.

Algorithm. 1) Choose a model pair $\tilde{L} = (\tilde{\ell}, \tilde{U})$.
2) Construct the matrices $r(x, \rho, \mu), S(\rho), \tilde{\varphi}(x, \rho)$.
3) Find $\varphi(x, \rho)$ by solving the main equation (3).
4) Construct $Q(x)$ and h.

Theorem 4. *For a matrix $M(\rho) \in \mathcal{M}$ to be the Weyl matrix for a pair $L = (\ell, U)$, it is necessary and sufficient that the following conditions are fulfilled:*
1) (asymptotics) there exists a pair $\tilde{L} = (\tilde{\ell}, \tilde{U})$ such that $M(\rho) - \tilde{M}(\rho) = O(\rho^{-1}),\ |\rho| \to \infty$, holds;
2) (condition P) for each fixed $x \geq 0$, the main equation (3) has a unique solution $\varphi(x, \rho)$ in the class $\varphi(x,\rho)D(x,\rho) \in \mathcal{B}_p$, $p > 1$, and $\sup\limits_{x \geq 0} \|\varphi(x,\rho)D(x,\rho)\|_{\mathcal{B}_p} < \infty$;
3) $\varepsilon(x) \in W$, where

$$\varepsilon(x) = \frac{1}{2\pi i} \int_\omega \Big(\Phi(x,\mu) A_0(\mu) \tilde{\Phi}^*(x,\mu) Q_0 - Q_0 \Phi(x,\mu) A_0(\mu) \tilde{\Phi}^*(x,\mu) \Big)\, d\mu.$$

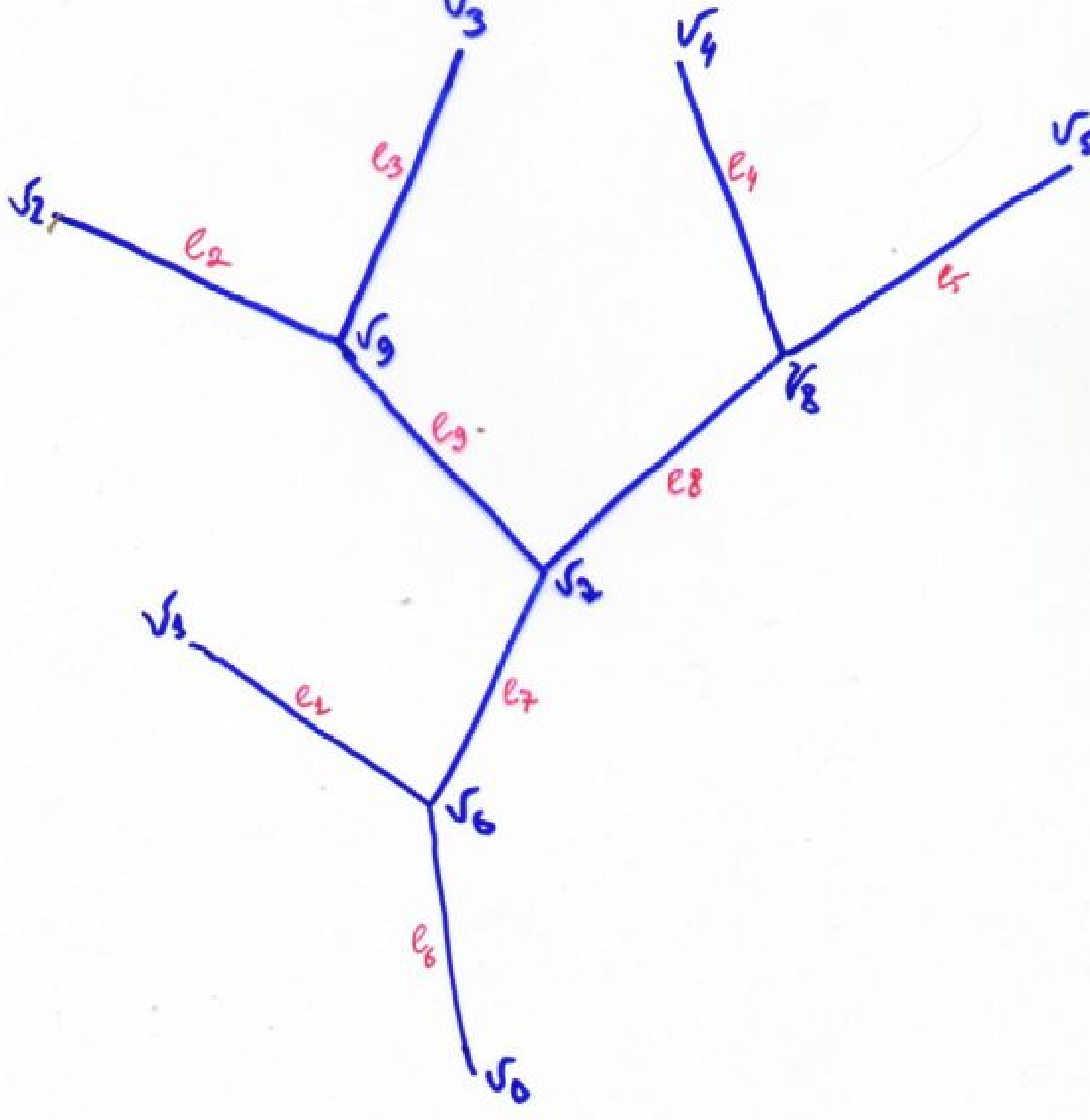
Under these conditions the pair $L = (\ell, U)$ is constructed by the formulae

$$Q(x) = \tilde{Q}(x) + \varepsilon(x), \quad h = \tilde{h}.$$

Inverse Spectral Problems for Sturm-Liouville Operators on Graphs

Yurko V.A. Inverse Problems, 21, no.3 (2005), 1075-1086.

Let T be a compact, simply connected rooted tree in $\mathbf{R}^{\mathbf{m}}$ with the root v_0, the set of vertices $V = \{v_0, \ldots, v_r\}$ and the set of edges $\mathcal{E} = \{e_1, \ldots, e_r\}$. We suppose that the length of each edge is equal to 1. For two points $a, b \in T$ we will write $a \leq b$ if a lies on a unique simple path connecting the root v_0 with b; let $|b|$ stand for the length of this path. We will write $a < b$ if $a \leq b$ and $a \neq b$. If $a < b$ we denote $[a, b] := \{z \in T : \ a \leq z \leq b\}$. If $e = [v, w]$ is an edge, we call v its initial point, w its end point and say that e emanates from v and terminates at w. We denote by $R(v) := \{e \in \mathcal{E} : \ e = [v, w], \ w \in V\}$ the set of edges emanating from v.



$V = \{v_0, \ldots, v_9\}$ are vertices, i.e. $n = 9$.

$\Gamma = \{v_0, \ldots, v_5\}$ are boundary vertices, i.e. $p = 5$

$\sigma = 4$ is the height of the tree

For any $v \in V$ the number $|v|$ is called the order of v. For $e \in \mathcal{E}$ its order is defined as the order of its end point. The number $\sigma := \max_{j=\overline{1,r}} |v_j|$ is called the height of the tree T. Let $V^{(\mu)} := \{v \in V : \ |v| = \mu\}$, $\mu = \overline{0, \sigma}$ be the set of vertices of order μ, and let
$\mathcal{E}^{(\mu)} := \{e \in \mathcal{E} : \ e = [v, w], \ v \in V^{(\mu-1)}, w \in V^{(\mu)}\}$, $\mu = \overline{1, \sigma}$ be the set of edges of order μ. Each edge $e \in \mathcal{E}$ is viewed as a segment $[0, 1]$ and is parameterized by the parameter $x \in [0, 1]$. We choose the following orientation on each edge $e = [v, w] \in \mathcal{E}$: if $z = z(x) \in e$, then $z(0) = w$, $z(1) = v$, i.e. $x = 0$ corresponds to the end point w. We enumerate the vertices v_j as follows: $\Gamma := \{v_0, v_1, \dots, v_p\}$ are boundary vertices, $v_{p+1} \in V^{(1)}$, and v_j, $j > p + 1$ are enumerated in order of increasing $|v_j|$. We enumerate the edges similarly, namely: $e_j = [v_{j_k}, v_j]$, $j = \overline{1, r}$, $j_k < j$. In particular, $E := \{e_1, \dots, e_{p+1}\}$ is the set of boundary edges; $e_{p+1} = [v_0, v_{p+1}]$ is called the rooted edge of T.

An integrable function Y on T may be represented as a vector $Y(x) = [y_j(x)]_{j\in J}$, $x \in [0,1]$, where $J := \{j : j = \overline{1, r}\}$, and the function $y_j(x)$ is defined on the edge e_j. Let $q = [q_j]_{j\in J}$ is an integrable real-valued function on T which is called the potential. Consider the Sturm-Liouville equation on T:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad x \in [0,1],\ j \in J, \tag{1}$$

$y_j(x), y_j'(x) \in AC[0,1]$ and satisfy the following $2r - p - 1$ matching conditions in each internal vertex v_k, $k = \overline{p+1, r}$:

$$y_j(1) = y_k(0) \text{ for all } e_j \in R(v_k), \quad \sum_{e_j \in R(v_k)} y_j'(1) = y_k'(0). \tag{2}$$

Let L_0 be the boundary value problem (BVP) defined by (1)-(3), where

$$Y_{|v_j} = 0, \quad j = \overline{0, p}. \tag{3}$$

Let L_k, $k = \overline{1, p}$, be the BVP for (1) satisfying (2) and

$$y_k'(0) = 0, \quad Y_{|v_j} = 0, \quad j = \overline{0, p} \setminus k. \qquad (4)$$

Let $\{\lambda_{lk}\}_{l \geq 1}$, $k = \overline{0, p}$, be the eigenvalues of L_k of the form (1)-(3) for $k = 0$, and (1), (2), (4) for $k = \overline{1, p}$, respectively.
Let $\Psi_k(x, \lambda) = [\psi_{kj}(x, \lambda)]_{j \in J}$, $k = \overline{0, p}$, be solutions of equation (1) satisfying (2) and the boundary conditions

$$\Psi_{k\,|v_j} = \delta_{kj}, \quad j = \overline{0, p}. \qquad (5)$$

Let $M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}}$, $M_k(\lambda) := \psi_{kk}'(0, \lambda)$.

Inverse Problem 1. Given the Weyl vector M, construct q on T.

Inverse Problem 2. Given spectra $\{\lambda_{lk}\}_{l \geq 1}$, $k = \overline{0, p}$, construct q on T.

The Weyl functions $M_k(\lambda)$ are meromorphic in λ:

$$M_k(\lambda) = \frac{\Delta_k(\lambda)}{\Delta(\lambda)}, \quad k = \overline{1, p},$$

where $\Delta(\lambda)$ and $\Delta_k(\lambda)$ are the characteristic functions for L_0 and L_k, respectively.

Let α_{lk} be the residues of $M_k(\lambda)$ at the poles λ_{l0}.

The data $S := \{\lambda_{l0}, \alpha_{lk}\}_{l\geq 1,\, k=\overline{1,p}}$ are called the spectral data for L_0.

Inverse Problem 3. Given S, construct the potential q on T.

Local inverse problem. Fix $k = \overline{1, p}$, and consider the following auxiliary inverse problem on the edge e_k:

IP(k). Given $M_k(\lambda)$, construct $q_k(x)$, $x \in [0, 1]$.

Theorem 1. *If $M_k(\lambda) = \tilde{M}_k(\lambda)$, then $q_k(x) = \tilde{q}_k(x)$ a.e. on [0,1]. Thus, the specification of the Weyl function M_k uniquely determines the potential q_k on the edge e_k.*

Using the method of spectral mappings for the Sturm-Liouville operator on the edge e_k one can get a constructive procedure for the solution of the local inverse problem IP(k).

Pseudo-cutting procedure!

Let $C_j(x,\lambda)$, $S_j(x,\lambda)$, $j \in J$, $x \in [0,1]$ be solutions of Eq. (1) on the edge e_j under the conditions $C_j(0,\lambda) = S_j'(0,\lambda) = 1$, $C_j'(0,\lambda) = S_j(0,\lambda) = 0$. Denote

$$M_{kj}^0(\lambda) = \psi_{kj}'(0,\lambda),\ M_{kj}^1(\lambda) = \psi_{kj}(0,\lambda),\ k = \overline{0,p},\ j \in J.$$

Then

$$\psi_{kj}(x,\lambda) = M_{kj}^1(\lambda) C_j(x,\lambda) + M_{kj}^0(\lambda) S_j(x,\lambda), \tag{6}$$

$$\psi_{kk}(x,\lambda) = C_k(x,\lambda) + M_k(\lambda) S_k(x,\lambda), \quad k = \overline{1,p}. \tag{7}$$

Problem Z(T, v₀, a). Let $\Psi = [\psi_j]_{j \in J}$ be the solution of equation (1) on T satisfying (2) and the boundary conditions

$$\Psi\Big|_{v_j} = a\delta_{j0}, \quad v_j \in \Gamma, \ a \in \mathbf{C}. \tag{8}$$

Denote $m_j^0(\lambda) = \psi_j'(0, \lambda)$, $m_j^1(\lambda) = \psi_j(0, \lambda)$, $j \in J$. Then

$$\psi_j(x, \lambda) = m_j^1(\lambda) C_j(x, \lambda) + m_j^0(\lambda) S_j(x, \lambda). \tag{9}$$

Substituting (9) into (2) and (8) we obtain a linear algebraic system for $m_j^0(\lambda), m_j^1(\lambda)$, $j \in J$. The determinant of this system is $\Delta(\lambda)$. Solving this system by Kramer's rule we find the transition matrix $[m_j^0(\lambda), m_j^1(\lambda)]_{j \in J}$ for T with respect to v_0 and a.

Fix $v_k \in V$. Denote $T_k^0 := \{z \in T : \; v_k < z\}$, $T_k := T \setminus T_k^0$. Let Γ_k be the set of boundary vertices of T_k. Denote $J_k := \{j : \; e_j \in T_k\}$. Fix $v_k \notin \Gamma$. Let $\Psi_k(x,\lambda) = [\psi_{kj}(x,\lambda)]_{j\in J_k}$ be the solution of (1)-(2), $\Psi_{k|v_j} = \delta_{kj}$ on T_k, $v_j \in \Gamma_k$; $M_k(\lambda) := \psi'_{kk}(0,\lambda)$, $k = \overline{p+1,r}$ is the WF on T_k for v_k.

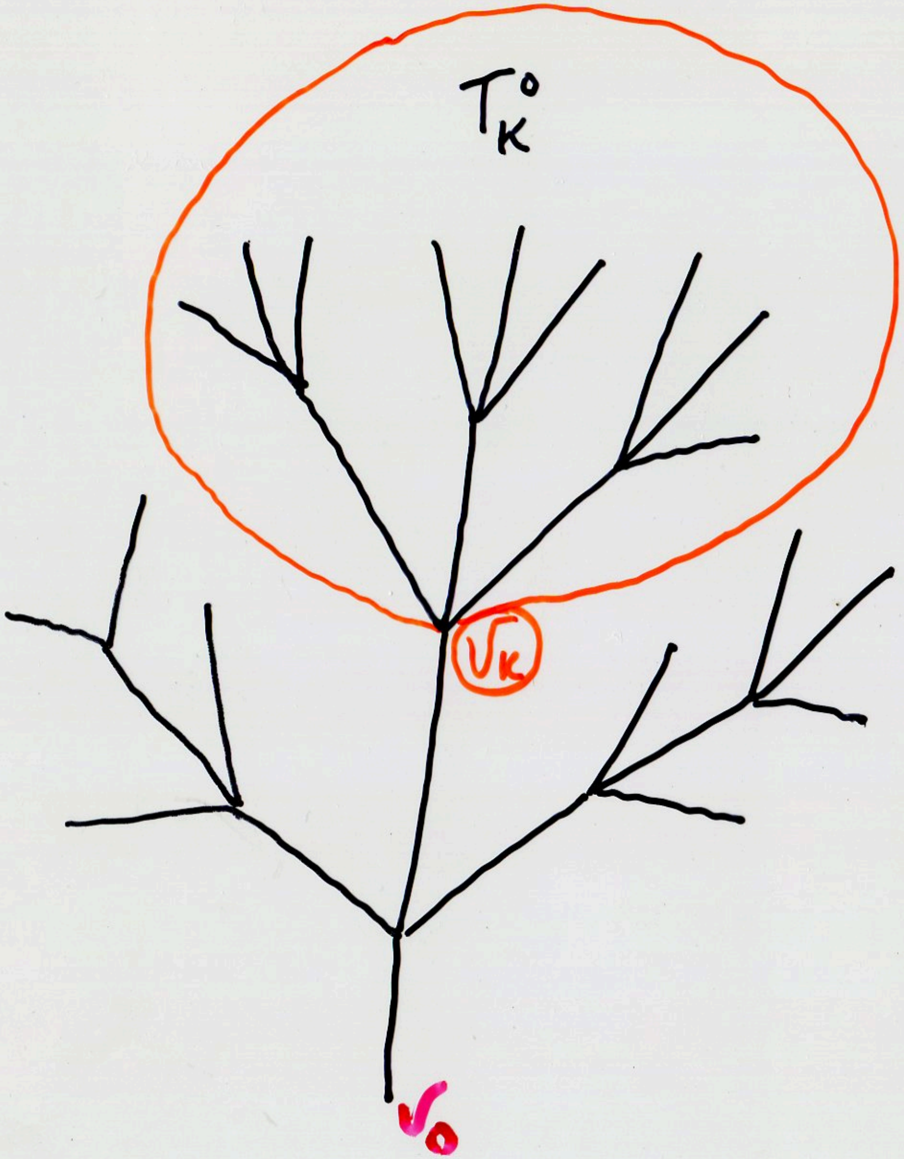
Lemma. *Fix $v_m \notin \Gamma$. Let $e_k = [v_m, v_k] \in R(v_m)$. Then*

$$M_m(\lambda) = \frac{1}{\psi_{kk}(1,\lambda)} \sum_{e_j \in R(v_m)} \psi'_{kj}(1,\lambda). \tag{10}$$

Denote $M_{kj}^0(\lambda) = \psi'_{kj}(0,\lambda)$, $M_{kj}^1(\lambda) = \psi_{kj}(0,\lambda)$, $k = \overline{p+1,r}$, $j \in J_k$. Then (6) and (7) hold for $k = \overline{1,r}$, $j \in J_k$, where $J_k = J$ for $k = \overline{1,p}$ $\rightarrow$

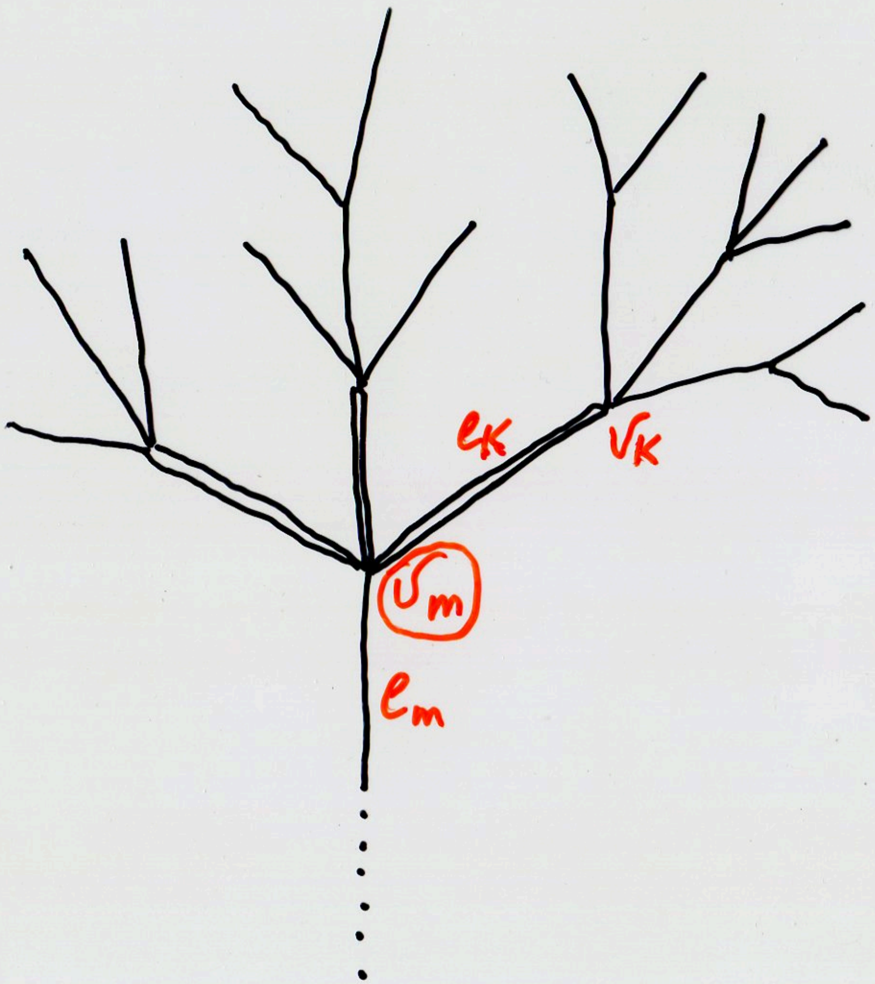
$$\psi_{kj}^{(\nu)}(1,\lambda) = M_{kj}^1(\lambda) C_j^{(\nu)}(1,\lambda) + M_{kj}^0(\lambda) S_j^{(\nu)}(1,\lambda), \tag{11}$$

$$\psi_{kk}^{(\nu)}(1,\lambda) = C_k^{(\nu)}(1,\lambda) + M_k(\lambda) S_k^{(\nu)}(1,\lambda). \tag{12}$$



$$T = T_K \cup T_K^{\circ}$$

$$M_m(\lambda) = \frac{1}{\psi_{KK}(l,\lambda)} \sum_{e_j \in R(v_m)} \psi'_{Kj}(l,\lambda)$$



Solution of Inverse Problems 1-3. Let us formulate the uniqueness theorems for the solution of these inverse problems.

Theorem 2. *The specification of the Weyl vector M uniquely determines the potential q on T.*

Theorem 3. *The specification of the spectra $\{\lambda_{lk}\}_{l\geq 1}$ of the boundary value problems L_k, $k=\overline{0,p}$ uniquely determines the potential q on T.*

Theorem 4. *The specification of the spectral data S uniquely determines the potential q on T.*

Let the Weyl vector $M(\lambda)=[M_k(\lambda)]_{k=\overline{1,p}}$ for the tree T be given. The procedure for the solution of Inverse Problem 1 consists in the realization of the so-called A_μ- procedures successively for $\mu=\sigma,\sigma-1,\dots,1$, where σ is the height of the tree T.

A_σ- procedure. 1) For each edge $e_k \in \mathcal{E}^{(\sigma)}$, we solve the local inverse problem IP(k) and find $q_k(x)$, $x \in [0,1]$ on the edge e_k.

2) For each $e_k \in \mathcal{E}^{(\sigma)}$, we construct $C_k(x,\lambda), S_k(x,\lambda)$, and calculate $\psi_{kk}^{(\nu)}(1,\lambda)$, $\nu = 0,1$, by (12): $\psi_{kk}^{(\nu)}(1,\lambda) = C_k^{(\nu)}(1,\lambda) + M_k(\lambda)S_k^{(\nu)}(1,\lambda)$.

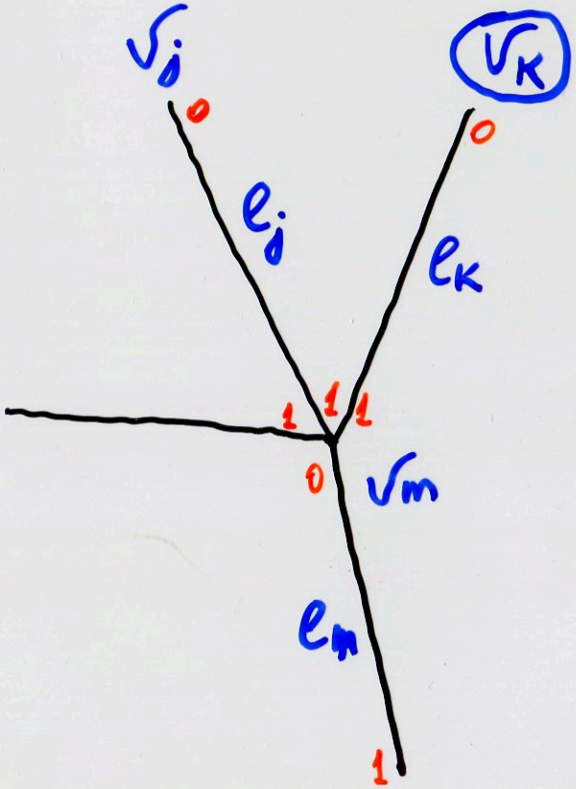
3) Returning procedure. For each fixed $v_m \in V^{(\sigma-1)} \setminus \Gamma$ and for all $e_j, e_k \in R(v_m)$, $j \neq k$, we construct $M_{kj}^s(\lambda)$, $s = 0,1$, via

$$M_{kj}^1(\lambda) = 0,\ M_{kj}^0(\lambda) = \psi_{kk}(1,\lambda)/S_j(1,\lambda),\ e_j, e_k \in R(v_m),\ j \neq k.$$

4) For each $v_m \in V^{(\sigma-1)} \setminus \Gamma$ we find $M_m(\lambda)$ by (10), where $\psi'_{kj}(1,\lambda)$ are constructed via (11): $\psi'_{kj}(1,\lambda) = M_{kj}^1(\lambda)C_j'(1,\lambda) + M_{kj}^0(\lambda)S_j'(1,\lambda)$.

Now we carry out A_μ- procedures for $\mu = \overline{1, \sigma-1}$ by induction. Fix $\mu = \overline{1, \sigma-1}$, and suppose that $A_\sigma, \dots, A_{\mu+1}$- procedures have been already carried out. Let us carry out A_μ- procedure.

A_σ-procedure



$v_m \in V^{(\sigma-1)} \setminus \Gamma$. Fix v_K.

Consider Ψ_K on e_j $(j \neq K)$.

$$\psi_{Kj}(0,\lambda) = 0, \quad \psi_{Kj}(1,\lambda) = \psi_{KK}(1,\lambda),$$

$$\psi_{Kj}(x,\lambda) = M^1_{Kj}(\lambda)\, C_j(x,\lambda) + M^0_{Kj}(\lambda)\, S_j(x,\lambda) \rightarrow$$

$$\boxed{M^1_{Kj}(\lambda) = 0, \; j \neq K} \rightarrow$$

$$\psi_{Kj}(1,\lambda) = M^0_{Kj}(\lambda)\, S_j(1,\lambda) \rightarrow$$

$$\boxed{M^0_{Kj}(\lambda) = \frac{\psi_{KK}(1,\lambda)}{S_j(1,\lambda)}, \quad j \neq K}$$

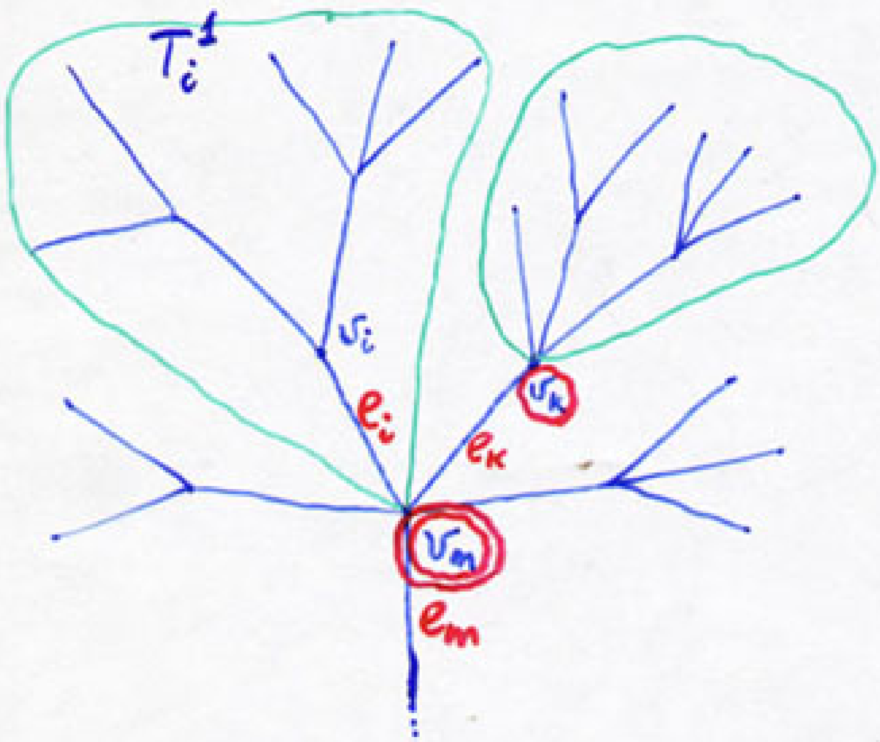
Returning procedure

Let $v_m \in V^{(\mu-1)} \setminus \Gamma$.

Fix $e_k \in R(v_m)$ and consider $\Psi_k = [\psi_{kj}]_{j \in J_k}$ (constructed for T_k) on T_i^1

Solving the problem $Z(T_i^1, v_m, \psi_{kk}(1,\lambda))$ we calculate $[M_{kj}^0(\lambda), M_{kj}^1(\lambda)]$ for $e_j \in T_i^1$.

Thus, we get $\psi_{kj}(x,\lambda)$ for $e_j \in T_i^1$



$\mathbf{A}_\mu$- procedure. For each $v_k \in V^{(\mu)}$, the functions $M_k(\lambda)$ are given. Indeed, if $v_k \in V^{(\mu)} \cap \Gamma$, then $M_k(\lambda)$ are given a priori, and if $v_k \in V^{(\mu)} \setminus \Gamma$, then $M_k(\lambda)$ were calculated on the previous steps.

1) For each edge $e_k \in \mathcal{E}^{(\mu)}$, we solve IP(k) and find $q_k(x)$ on e_k. If $\mu = 1$, then Inverse Problem 1 is solved. If $\mu > 1$, we go on to the next step.

2) For each $e_k \in \mathcal{E}^{(\mu)}$, we construct $C_k(x,\lambda), S_k(x,\lambda)$, and calculate $\psi_{kk}^{(\nu)}(1,\lambda)$, $\nu = 0, 1$, by (12): $\psi_{kk}^{(\nu)}(1,\lambda) = C_k^{(\nu)}(1,\lambda) + M_k(\lambda) S_k^{(\nu)}(1,\lambda)$.

3) Returning procedure. For each fixed $v_m \in V^{(\mu-1)} \setminus \Gamma$ and for any fixed $e_k, e_i \in R(v_m)$, $i \neq k$, we consider the tree $T_i^1 := T_i^0 \cup \{e_i\}$ with the root v_m. Solving the problem $Z(T_i^1, v_m, \psi_{kk}(1,\lambda))$, we calculate the transition matrix $[M_{kj}^0(\lambda), M_{kj}^1(\lambda)]$ for $e_j \in T_i^1$.

4) For each fixed $v_m \in V^{(\mu-1)} \setminus \Gamma$ we calculate the Weyl function $M_m(\lambda)$ by (10), where $\psi'_{kj}(1,\lambda)$ are constructed via (11) for $\nu = 1$.

1) Sturm-Liouville operators on arbitrary compact graphs.
2) Sturm-Liouville operators on noncompact graphs.
3) Higher order differential operators on graphs.
4) Variable order differential operators on graphs.
5) Pencils of differential operators on graphs.
6) Differential operators with singularities on graphs.