The Bernstein-Landau paradox in fusion plasmas. An operator theory point of view

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To Dimitri (Dima) Yafaev In Memoriam

This talk is based on the paper,

Fréderique Charles, Bruno Després, Alexandre Regge, Ricardo Weder, The Magnetized Vlasov-Ampère System and the Bernstein-Landau Paradox, J. Stat. Phys. 183 23, 2021, 57 pp.

The Vlasov-Poisson system for a plasma

We approximate the Newton equation for a very large number of charged particles moving in an electromagnetic field by a density function $f(t,x,v) \geq 0$. We assume that x is the position of the particles and that the motion is one dimensional along the first coordinate axis. The velocity, v of the particles is two dimensional, $v = (v_1, v_2) \in \mathbb{R}^2$. The density function is a solution to the following Vlasov equation,

$$\partial_t f + v_1 \partial_x f + \mathbf{F} \cdot \nabla_v f = 0.$$

We assume that the motion of the charged particles is a 2π-periodic oscillation. We look for solutions $f(t, x, v)$, for $t \in \mathbb{R}, x \in [0, 2\pi], v = (v_1, v_2) \in \mathbb{R}^2$, that are periodic in x, i.e., $f(t, 0, v) = f(t, 2\pi, v)$.

The electromagnetic Lorentz force is given by

$$\mathbf{F}(t, x) = \frac{q}{m} (\mathbf{E}(t, x) + v \times \mathbf{B}(t, \mathbf{x})),$$

We suppose that the magnetic field $\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_0$ is constant in space-time, and that the two dimensional velocity v is perpendicular to the constant magnetic field, i.e., $\mathbf{B}_0 = (0, 0, B_0), B_0 > 0$. Moreover, we assume that the electric field is directed along the first coordinate axis, $\mathbf{E}(t, x) = (E(t, x), 0, 0)$.

We adopt a convenient normalization adapted to electrons, that is $q_{\mathrm{ref}} = -1$ and $m_{\mathrm{ref}} = 1$, where q_{ref} is the charge of the electron, and m_{ref} is the mass of the electron.

The electric field satisfies the Gauss law,

$$\partial_x E(t, x) = 2\pi - \int_{\mathbb{R}^2} f dv,$$

where 2π is the constant density of the heavy ions, that do not move, and that we take equal to 2π for convenience. The term

$$-\int_{\mathbb{R}^2} f \, dv$$

is the charge density of the particles with charge -1.

With these notations and normalizations we obtain the following

Vlasov-Poisson system,

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) f = 0, \\ \partial_x E(t,x) = 2\pi - \int_{\mathbb{R}^2} f dv. \end{cases}$$

We denote the cyclotron frequency by $\omega_{\mathrm{c}} := B_0$.
We retain the potential part of the electric field

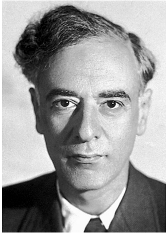
$$E(t,x) = -\partial_x \varphi(t,x),$$

where the potential $\varphi(t,x)$ is a solution to the Poisson equation,

$$-\Delta\varphi = 2\pi - \int_{\mathbb{R}^2} f dv.$$

The potential is assumed to be periodic, $\varphi(t,0) = \varphi(t,2\pi)$,

The question that we address is: ? What is the long time behaviour of the electric field?
Suppose that there is no magnetic field $B_0 = 0$.
Landau Damping. L. Landau, J. Phys. USSR **10** n^0 25 (1946).



$$\lim_{t\to\infty} E(x,t) = 0.$$

This is a remarkable fundamental fact in plasma physics.
For large times all the energy of the electric field is transferred to the electrons. There is no dissipation mechanism
The Landau damping has been extensively studied. Both in the linear and nonlinear cases.
C. Mouhot and C. Villani Acta Math. **207** 29-201 (2011)

There is currently a lot of work being done mathematicians, mathematical physicists and numerical analysts in the Vlasov-Poisson equations, and other kinetic equations, in the context of Landau damping and related fields.
An important motivation is the ITER project www.iter.org, a large Tokamak that is been built in the south of France.

You will hear more about Landau damping in the talk by B. Després Friday.

However, what happens when the magnetic field $\mathbf{B}_0$ is different from zero

Recall the Vlasov-Poisson system with magnetic field

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) f = 0, \\ \partial_x E(t,x) = 2\pi - \int_{\mathbb{R}^2} f dv. \end{cases}$$

$$E(t,x) = -\partial_x \varphi(t,x),$$

$$-\Delta\varphi = 2\pi - \int_{\mathbb{R}^2} f dv.$$

$$\int_0^{2\pi} E(t,x)\, dx = 0.$$

The Bernstein-Landau paradox

The Landau damping disappears. The electric field is oscillatory in time

I. B. Bernstein. Phys. Rev. **109** 10-21 (1958)

The terminology Bernstein-Landau paradox was coined by A. I. Sukhorukov and P. Stubbe, On the Bernstein-Landau paradox, Physics of Plasmas **4**, 2497, 1997.

Linearization

We linearize the equations around a homogeneous Maxwellian equilibrium state $f_0(v)$, where,

$$f_0(v) := e^{\frac{-v^2}{2}}.$$

It corresponds to the expansion

$$f(t,x,v) = f_0(v) + \varepsilon\sqrt{f_0(v)}u(t,x,v) + O(\varepsilon^2),$$

and

$$E(t,x) = E_0 + \varepsilon F(t,x) + O(\varepsilon^2),$$

with a null reference electric field $E_0 = 0$.

Keeping the terms up to linear in ε, one gets the linearized magnetized Vlasov-Poisson system written as,

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 \sqrt{f_0} + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) u = 0, \\ \partial_x F = - \displaystyle\int_{\mathbb{R}^2} u \sqrt{f_0} dv, \\ \displaystyle\int_{[0,2\pi]} F = 0, \end{cases}$$

where in the third equation we have added the constraint that the mean value of the electric field F is zero, as before.

Moreover, the electric field $F(t,x)$, is obtained from a potential,

$$F(t,x) = -\partial_x \varphi(t,x),$$

where the potential is periodic, $\varphi(t,0) = \varphi(t,2\pi)$, and it solves the Poisson equation,

$$-\Delta \varphi = - \int_{\mathbb{R}^2} u \sqrt{f_0} dv.$$

The study of the solutions to the magnetized Vlasov-Poisson system is the standard method to analyze the dynamics of a very large number of charged particles moving in the presence of a constant external magnetic field. We now present an alternate method to study this problem.

In the full Maxwell equations one of the equation is the Ampère equation

$$\partial_t F = \int_{\mathbb{R}^2} v_1 \, u \sqrt{f_0} dv.$$

We consider here the following modified Ampère equation

$$\partial_t F = I^* \int_{\mathbb{R}^2} v_1 \sqrt{f_0} \, u \, dv,$$

where I^* is given by

$$I^* g(x) := g(x) - \frac{1}{2\pi} \int_0^{2\pi} g(y) \, dy.$$

With this convention the magnetized Vlasov-Ampère system is written as follows,

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 \sqrt{f_0} + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) u = 0, \\ \partial_t F = I^* \int_{\mathbb{R}^2} v_1 \sqrt{f_0}\, u dv. \end{cases}$$

To the magnetized Vlasov-Ampère system we add following conditions for $F_{\text{ini}} := F(0, \cdot)$ and $u_{\text{ini}} = u(0, \cdot, \cdot)$: the integral constraint,

$$\int_0^{2\pi} F_{\text{ini}}\, dx = 0,$$

is satisfied at initial time, and the Gauss law is also satisfied at the initial time,

$$\frac{d}{dx} F_{\text{ini}} = -\int_{\mathbb{R}^2} u_{\text{ini}} \sqrt{f_0} dv.$$

Lemma

The linearized magnetized Vlasov-Poisson system is equivalent to the magnetized Vlasov-Ampère system with the condition $\int_0^{2\pi} F_{\text{ini}}\, dx = 0$, and the Gauss law being satisfied initially.

A fundamental energy relation is easily shown for solutions of the magnetized Vlasov-Ampère system,

$$\frac{d}{dt}\left(\int_{[0,2\pi]\times\mathbb{R}^2} \frac{u^2}{2} dxdv + \int_{[0,2\pi]} \frac{F^2}{2} dx\right) = 0.$$

The term

$\displaystyle\int_{[0,2\pi]\times\mathbb{R}^2} \int_{\mathbf{v}} \frac{u^2}{2} dxdv$, is the kinetic energy of the particles, and the term

$$\int_{[0,2\pi]} \frac{F^2}{2} dx, \qquad \text{is the energy of the electric field.}$$

This identity, that expresses the conservation of the energy, is the basis of our formulation of the magnetized Vlasov-Ampère system as a Schrödinger equation in Hilbert space, where the magnetized the Vlasov-Ampère operator plays the role of the selfadjoint Hamiltonian.

We denote by $L^2(0,2\pi)$ the standard Hilbert space of functions that are square integrable in $(0,2\pi)$. Furthermore, we designate by $L^2_0(0,2\pi)$ the closed subspace of $L^2(0,2\pi)$ consisting of all functions with zero mean value, i.e.,

$$L^2_0(2,\pi) := \left\{ F \in L^2(0,2\pi) : \int_0^{2\pi} F(x)\, dx = 0 \right\}.$$

Further, we denote by $L^2(\mathbb{R}^2)$ the standard Hilbert space of all functions that are square integrable in $\mathbb{R}^2$. Let us denote by $\mathcal{A}$ the tensor product of $L^2(0,2\pi)$ and of $L^2(\mathbb{R}^2)$, namely,

$$\mathcal{A} := L^2(0,2\pi) \otimes L^2(\mathbb{R}^2).$$

Our space of physical states, that we denote by $\mathcal{H}$, is defined as the direct sum of $\mathcal{A}$ and $L^2_0(0, 2\pi)$.

$$\mathcal{H} := \mathcal{A} \oplus L^2_0(0, 2\pi).$$

We find it convenient to write $\mathcal{H}$ as the space of the column vector-valued functions,

$$\begin{pmatrix} u \\ F \end{pmatrix}$$

where $u(x, v) \in \mathcal{A}$ and $F(x) \in L^2_0(0, 2\pi)$. The scalar product in $\mathcal{H}$ is given by,

$$\left(\begin{pmatrix} u \\ F \end{pmatrix}, \begin{pmatrix} f \\ G \end{pmatrix} \right)_{\mathcal{H}} := (u, f)_{\mathcal{A}} + (F, G)_{L^2(0,2\pi)}.$$

Note that for the solutions of the magnetized Vlasov-Ampère system the $\mathcal{H}-$ norm is constant in time, as is is equal to twice the energy.

We write the magnetized Vlasov-Ampère system as a Schrödinger equation in the Hilbert space $\mathcal{H}$ as follows

$$i\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = \mathbf{H} \begin{pmatrix} u \\ F \end{pmatrix},$$

where the magnetized Vlasov-Ampère operator **H** is the following operator in $\mathcal{H}$,

$$\mathbf{H} = \begin{bmatrix} H_0 & -iv_1\, e^{\frac{-v^2}{4}} \\ iI^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}} \cdot dv & 0 \end{bmatrix},$$

with

$$H_0 = i\left(-v_1 \partial_x + \omega_{\mathrm{c}}(v_2\, \partial_{v_1} - v_1\, \partial_{v_2})\right), \qquad \omega_c := B_0.$$

write $\mathbf{H}$ in the following form,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{V},$$

where

$$\mathbf{H}_0 := \begin{bmatrix} H_0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\mathbf{V} := \begin{bmatrix} 0 & -iv_1\, e^{\frac{-v^2}{4}} \\ iI^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}} \cdot dv & 0 \end{bmatrix}.$$

The domain of **H** is defined as follows,

$$D[\mathbf{H}] := D[H_0] \oplus L^2_0(0, 2\pi).$$

For a precise definition of the domain of H_0 in $\mathcal{A}$ see our paper in J. Stat. Phys.. For the purpose of this talk it is enough to say that the functions $u \in D[H_0] \subset \mathcal{A}$ have to satisfy,

$$i\,(-v_1\partial_x + \omega_c(v_2\,\partial_{v_1} - v_1\,\partial_{v_2}))\,u \in \mathcal{A}.$$

H is selfadjoint in $\mathcal{H}$, and it has pure point spectrum. We prove this by explicitly computing the eigenvalues and a orthonormal basis of eigenfunctions. And also by an abstract operator theoretical argument.

This explains the Bernstein-Landau paradox.
The general solution is a linear combination of time-independent solutions (for the eigenvalue zero) and of oscillatory solutions (for the non-zero eigenvalues).
Note that **H** is formally analytic in $\omega_c := B_0$. However, the domain of **H** changes abruptly when $\omega_c := B_0 = 0$.
For $\omega_c \neq 0$ we requiere

$$i\,(-v_1\partial_x + \omega_{\rm c}(v_2\,\partial_{v_1} - v_1\,\partial_{v_2}))\,u \in \mathcal{A}.$$

and for $\omega_c = 0$,

$$i\,(-v_1\partial_x +)\,u \in \mathcal{A}.$$

It is a well known fact that the spectrum of a family of linear operators depending on a parameter can change abruptly when the domain changes for some value of the parameter.

A simple example is the harmonic oscillator,

$$-\Delta + \omega^2 x^2$$

that has pure point spectrum for $\omega \neq 0$, and no eigenvalues, and continuous spectrum $[0, \infty)$ for $\omega = 0$.
So the Bernstein-Landau paradox is just a well known fact of the perturbation theory of families of selfadjoint operators in Hilbert space.
Then, our problem is to compute the eigenvalues of **H** and to prove that **H** has a complete orthonormal basis of eigenvectors. We do this in two different ways.

- By an explicit computations of the eigenvalues and by explicitly constructing an orthonormal basis of eigenvectors. This results not only is useful in the analysis of the Bernstein-Landau paradox, but it is also the basis for the numerical evaluation of the solution, as we will show, and it can be used for other purposes.

- By an abstract operator theoretical argument that proves that the spectrum is pure point and that there exists an orthonormal basis of eigenvectors, but that it does not gives us explicitly the eigenvectors.

This result is enough to prove that the Bernstein-Landau paradox exists. Further, it is much simpler, and it show how abstract operator theoretical methods can give simple solutions to problems in physics.

It what follows I discuss the first method.

The kernel of **H**

H has an infinite dimensional kernel with a complete basis of orthonormal eigenvectors with eigenvalue zero that we denote by

$$\left\{ \mathbf{V}_n^{(0)}, n \in \mathbb{Z}^* \right\}, \left\{ \mathbf{M}_{0,j}^{(0)}, j \in \mathbb{N}^* \right\},$$

$$\left\{ \mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j = 2, \dots \right\}, \left\{ \mathbf{F}_n^{(0)}, n \in \mathbb{Z}^* \right\},$$

where $\mathbb{N}^*$ are the positive integers and $\mathbb{Z}^*$ are the nonzero integers.

In our paper in J. Stat. Phys. we give the explicit formulae of these eigenvectors

The eigenvectors of **H** with eigenvalue zero allow us to express the Gauss law in a convenient way as an orthogonality relation in $\mathcal{H}$.

We define,

$$\mathcal{H}_{\rm G} := {\rm Span}\left[\left\{\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*\right\} \cup \mathbf{M}_{0,1}^{(0)}\right].$$

Then, $(u,F)^T \in \mathcal{H}$ satisfies the Gauss law in and only if $(u,F)^T \in \mathcal{H}_G^{\perp}$, that is to say, if and only if

$$\left(\begin{pmatrix} u \\ F \end{pmatrix}, \begin{pmatrix} m \\ J \end{pmatrix}\right)_{\mathcal{H}} = 0, \qquad \begin{pmatrix} m \\ J \end{pmatrix} \in \mathcal{H}_G.$$

In other problems, for example, for the Maxwell equations in wave guides, the Gauss law is equivalent to being orthogonal to the full kernel of the Mawell operator, not to a subspace of it , as it is the case for the magnetized Vlasov-Ampère operator. For this point see

R. Weder, *Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media,* Applied Mathematical Sciences **87**, Springer-Verlag, New York, 1991.

The Nonzero Eigenvalues

The following set of eigenfunctions of **H** with eigenvalue different from zero are explicitly computed in our paper in J. Stat. Phys..

$$\left\{\mathbf{V}_{m,j}, m \in \mathbb{Z}^*, j \in \mathbb{N}^*\right\}, \left\{\mathbf{W}_{n,m,j}, n, m \in \mathbb{Z}^*, j \in \mathbb{N}^*\right\}, \{\mathbf{Z}_{n,m}, n, m \in \mathbb{Z}^*\},$$

is a orthonormal basis of $\mathrm{Ker}[\mathbf{H}]^{\perp}$. Moreover,

$$\mathbf{V}_{m,j}, m \in \mathbb{Z}^*, j \in \mathbb{N}^* \text{ have eigenvalue, } \lambda_m^{(0)} = m\omega_c \neq 0,$$

and the eigenfunctions,

$$W_{n,m,j}, n, m \in \mathbb{Z}^*, j \in \mathbb{N}^* \text{ have eigenvalue, } \lambda_m^{(0)} = m\omega_c \neq 0.$$

The eigenfunctions

$$\mathbf{Z}_{n,m}, n, m \in \mathbb{Z}^*,$$

have eigenvalue $\lambda_{n,m} \in (m\,\omega_c, (m+1)\,\omega_c), \lambda_{n,m} \neq 0$, and moreover,

$$\lambda_{n,m} = m\omega_{\rm c} + 2\pi m\,\omega_{\rm c}\,\frac{a_{n,|m|}}{n^2} + a_{n,|m|}\,O\left(\frac{1}{|m|}\right), \qquad m \to \pm\infty.$$

where,

$$a_{n,m} := \int_0^\infty e^{\frac{-r^2}{2}}\, J_m\left(\frac{nr}{\omega_{\rm c}}\right)^2 r dr > 0, \quad m \in \mathbb{Z}^*.$$

This *Weyl estimate* plays a crucial role in the proof of the completeness of the eigenfunctions. The $\lambda_{n,m}$ are the Bernstein frequencies and the $\mathbf{Z}_{n,m}$ correspond to the Bernstein modes.

The General Solution

With the complete orthogonal system of eigenfunctions we can compute the general solution to the magnetized Vlasov-Ampère system. For simplicity we present the result for the charge density fluctuation,

$$\rho(t,x) := -\int_{\mathbb{R}^2} u(t,x,v)\,\sqrt{f_0}(v)dv.$$

We denote by $\mathbf{G}_0$ the initial data, with $\mathbf{G}_0 \in \mathcal{H}_G^{\perp}$.

$$\rho(t,x) = \rho_{\rm stat}(x) + \rho_{\rm din}(t,x),$$

where,

$$\rho_{\rm stat} := -\sum_{n\in\mathbb{Z}^*} \left(\mathbf{G}_0, \mathbf{F}_n^{(0)}\right) \int_{\mathbb{R}^2} \mathbf{F}^{(0,1)}(x,v)\, e^{\frac{-v^2}{4}}\, dv,$$

is the static part of the charge density fluctuation, and where $\mathbf{F}^{(0,1)}(x,v)$ is the first component of $\mathbf{F}_n^{(0)}$.

Moreover,

$$\rho_{\rm din}(t,x) = \sum_{n,m \in \mathbb{Z}^*} e^{-it\lambda_{n,m}} \, (\mathbf{G}_0, \mathbf{Z}_{n,m})_{\mathcal{H}} \, \rho_{n,m}(x),$$

is the time dependent part of the charge density fluctuation. Here $\rho_{n,m}(x)$ is the charge density fluctuation of the eigenfunction $\mathbf{Z}_{n,m}$.

The right-hand side is the expansion of the charge density fluctuation in the Bernstein modes. Note however, that for general initial data there is also the static part $\rho_{\rm stat}$ of the charge density fluctuation.

The existence of the static part, ρ_{stat} in general solution appears to be new, it is usually not reported in the physics literature. Further, it is also not reported in the mathematical literature.

In Theorem 1 of

J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker Planck equations in a uniform magnetic field, J. Stat. Phys. **178**, 552-594 (2020),

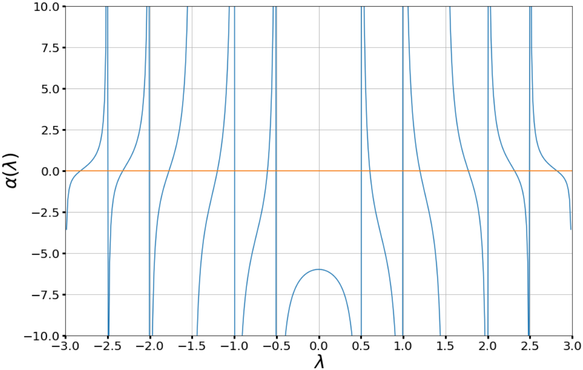
it is claimed that the static part ρ_{stat} of the charge density fluctuation is not present, i.e. that it is identically zero, and that there are only the oscillatory Bernstein modes. J. Bedrossian and F. Wang consider the problem in the case of three dimensions for the configuration and the velocity spaces. However, in Appendix 2 our paper in J. Stat. Phys. we give explicit examples that show that the static part is in general non zero for three dimensional configuration and velocity spaces.

Computation of the Eigenvalues

The eigenvalues $\lambda_{n,m}$ are the root of the secular function,

$$\alpha(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m \in \mathbb{Z}^*} \frac{m\omega_c}{m\omega_c + \lambda} a_{n,m}.$$

For $\omega_c = 0.5$ and $n = 1$.



$$\mathbf{U}(t) = e^{i\lambda_{n,m}t}\,\mathbf{U}_{\text{ini}}. \tag{1}$$

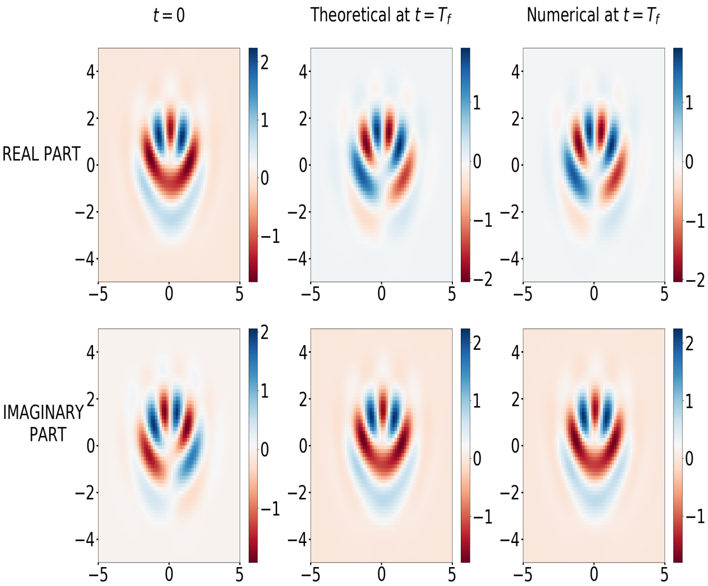
where $\mathbf{U}_{\text{ini}}$ is an eigenfunction of $\mathbf{H}$ with eigenvalue $-\lambda_{n,m} = \lambda_{m,-m}$.

In the following numerical results, we take $(n, m) = (1, 2)$, $\omega_c = 0.5$, and, $T_{\text{f}} = \frac{\pi}{2\lambda_{1,2}}$. This means that

$$\mathbf{U}(T_{\text{f}}) = \exp\left(i\frac{\pi}{2}\right)\mathbf{U}_{\text{ini}} = i\,\mathbf{U}_{\text{ini}},$$

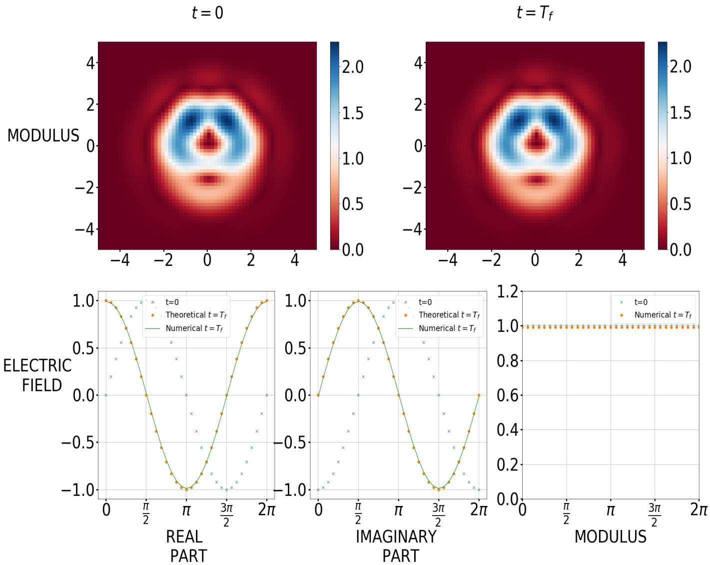
and then, the solution of the system at $t = T_{\text{f}}$ corresponds to the initial condition where the real (up to a sign) and imaginary parts have been exchanged.

We present the real and imaginary parts of the first component of $\mathbf{U}(t)$ in the $v_1 - v_2$ plane for $x = 0$.



t = 0
Theoretical at t = T_f
Numerical at t = T_f
REAL PART
IMAGINARY PART

Modulus of the first component of $\mathbf{U}(t)$ in $v_1 - v_2$ plane for $x = 0$, and the real and imaginary parts of F.



The Nonlinear Magnetized Vlasov-Poisson System

In the following pictures we show how the solution to the nonlinear magnetized Vlasov-Poisson system behaves when initialized with an eigenfunction of the magnetized Vlasov-Ampère system.

For the nonlinear magnetized Vlasov-Poisson system we initialize with

$$f_{ini} = f_0 + \varepsilon \sqrt{f_0}\, \mathrm{Re}(w_{n,m}), \qquad E_{\mathrm{ini}} = \varepsilon\, \mathrm{Re}(F_n),$$

where $w_{n,n}$ is the first component of the eigenfunction $\mathbf{U}_{\mathrm{ini}}$, and F_n is the second component. Further, ε is a scalar which controls the amplitude of the perturbation. We take $\varepsilon = 0.1$. As before, we take $(n, m) = (1, 2)$, $\omega_c = 0.5$,and, $T_{\mathrm{f}} = \frac{\pi}{2\lambda_{1,2}}$.

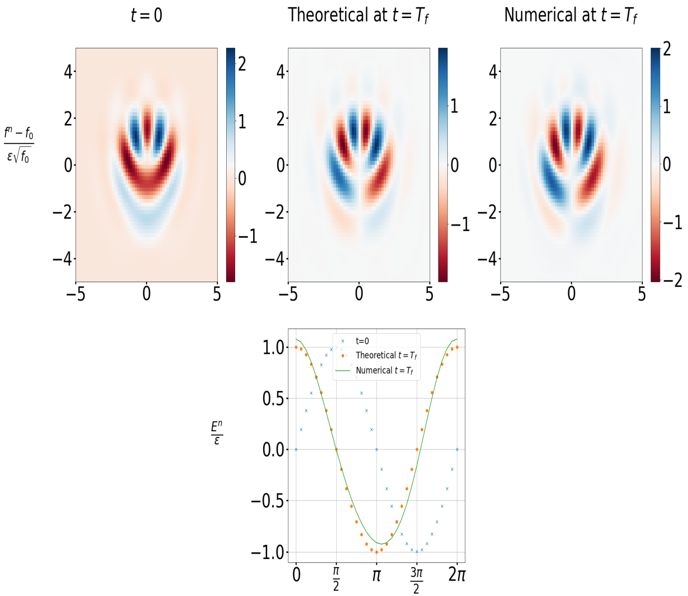
We compare the theoretical perturbation, u, F, that is given by

$$\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} = \mathrm{Re}(\mathbf{U}(t)),$$

with the numerical perturbation,

$$u = \frac{f - f_0}{\varepsilon \sqrt{f_0}}, \text{and } F = \frac{E}{\varepsilon}.$$

We plot u in $v_1 - v_2$ plane for $x = 0$ and electric field F.



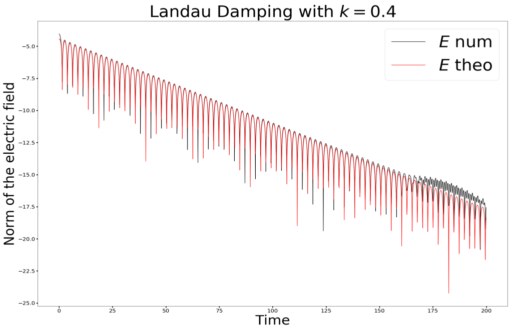
$t = 0$
Theoretical at $t = T_f$
Numerical at $t = T_f$
$\frac{f^n - f_0}{\varepsilon\sqrt{f_0}}$
$\frac{E^n}{\varepsilon}$
t=0
Theoretical $t = T_f$
Numerical $t = T_f$

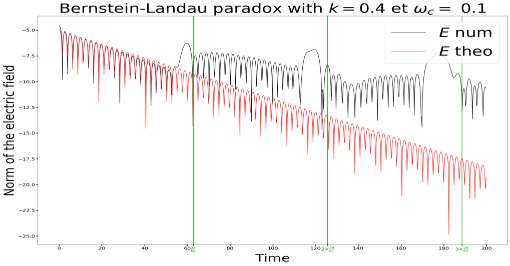
We numerically illustrate the Bernstein-Landau paradox and we compare it with Landau-Damping in the case $\omega_c = 0$.
We initialize the nonlinear magnetized Vlasov-Poisson system with the density function,

$$f_{LD}(x, v_1, v_2) = \frac{1}{2\pi} (1 + \varepsilon \cos kx)\ e^{\frac{-v^2}{2}}, \quad \varepsilon = 0.001, k = 0.4,$$

where we take $k = 0.4$. In this simulation the position interval is $[0, \frac{2\pi}{k}]$, since we keep periodic solutions.
We consider the approximate solution to the nonlinear Vlasov-Poisson system, with $\omega_c = 0$, given by E. Sonnendrücker, *Modèles Cinétiques pour la Fusion*, Notes du cours de M2. IRMA, Université L. Pasteur, Strasbourg, France, 2008.

$$E(x, t) \approx 4\varepsilon \times 0,424666 \exp(-0,0661 t) \sin(0,4x) \cos(1,2850t - 0,3357725).$$


Landau Damping with $k = 0.4$
E num
E theo
Norm of the electric field
Time


Bernstein-Landau paradox with $k = 0.4$ et $\omega_c = 0.1$
E num
E theo
Norm of the electric field
Time

Operator Theoretical Proof of the Bernstein-Landau Paradox

We introduce the following Hilbert spaces,

$$\mathcal{H}_0 := L^2(\mathbb{R}^2) \oplus \{0\},$$

$$\mathcal{H}_n := \mathrm{Span}\left[\frac{e^{inx}}{\sqrt{2\pi}}\right] \otimes \left(L^2(\mathbb{R}^2) \oplus \mathbb{C}\right), \qquad n \in \mathbb{Z}^*.$$

The functions $(u_n, \alpha_n)^T$ in $\mathcal{H}_n$ can be written as

$$\begin{pmatrix} u_n(x,v) \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \sum_{m\in\mathbb{Z}, j\in\mathbb{N}^*} u_{n,m,j}(x,v)\,(u_n, u_{n,m,j})_{\mathcal{A}} \\ \alpha_n \end{pmatrix},$$

where for $n = 0, \alpha_n = 0$. Here, $u_{n,m,j}$ is an orthonormal basis in $\mathcal{A}$, made of eigenfunctions of the Vlasov operator $H_0 := i\,(-v_1\partial_x + \omega_c(v_2\,\partial_{v_1} - v_1\,\partial_{v_2}))$.

Using the Fourier series in x we prove in our paper in J. Stat. Phys. that it is enough to prove that the following operators in $\mathcal{H}_n$ have pure point spectrum.

$$\mathbf{H}_n = \mathbf{H}_{0,n} + \mathbf{V}_n, \qquad n \in \mathbb{Z},$$

where $\mathbf{H}_{0,n}$ is the operator in $\mathcal{H}_n$ given by,

$$\mathbf{H}_{0,n} \begin{pmatrix} u_n(x,v) \\ \alpha_n \end{pmatrix} := \sum_{m\in\mathbb{Z}, j\in\mathbb{N}^*} \begin{pmatrix} \lambda_m^{(0)} u_{n,m,j}(x,v)\,(u_n, u_{n,m,j})_{\mathcal{A}} \\ 0 \end{pmatrix},$$

with domain

$$D[\mathbf{H}_{0,n}] := \{(u_n, \alpha_n)^T : \sum_{m\in\mathbb{Z}, j\in N} (\lambda_m^{(0)})^2 |(u_n, u_{n,m,j})_{\mathcal{A}}|^2 < \infty,$$

The spectrum of $\mathbf{H}_{0,n}$ is pure point and it consists of the infinite multiplicity eigenvalue $\lambda_m^{(0)}, m \in \mathbb{Z}$.

Recall that the discrete spectrum of a selfadjoint operator consists of the isolated eigenvalues of finite multiplicity, and that the essential spectrum is the complement in the spectrum of the discrete spectrum.

So, we have reached the conclusion that the spectrum of $\mathbf{H}_{0,n}$ coincides with the essential spectrum and it is given by the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$.

Further,

$$\mathbf{V}_n \begin{pmatrix} e^{inx}\tau(v) \\ \alpha_n \end{pmatrix} = e^{inx} \begin{pmatrix} -iv_1 e^{\frac{-v^2}{4}} \alpha_n \\ il^* \int_{\mathbb{R}^2} v_1 e^{\frac{-v^2}{4}} \tau(v)\, dv \end{pmatrix}.$$

The operator $\mathbf{V}_n$ is a rank two operator, hence, it is compact. Then, it is a consequence of the celebrated Weyl theorem for the invariance of the essential spectrum, that the essential spectrum of $\mathbf{H}_n, n \in \mathbb{Z}$ is given by the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$.

However, since the complement of the essential spectrum is discrete, we have that the spectrum of $\mathbf{H}_n$ consists of the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$, and of a set of isolated eigenvalues of finite multiplicity that can only accumulate at the essential spectrum and at $\pm\infty$.

We know from our previous results that these eigenvalues are the $\lambda_{n,m}, n, m \in \mathbb{Z}^*$, and that they are of multiplicity one. However, the operator theoretical argument does not tell us that.

However, it tells us that the spectrum of $\mathbf{H}_n$ is pure point and that $\mathbf{H}_n$ has a complete orthonormal set of eigenfunctions.

This proves that the Bernstein-Landau paradox.

THANKS FOR YOUR ATTENTION