The Marchenko method for the general system of derivative nonlinear Schrödinger equations

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ABSTRACT

Outline

The general DNLS system

- DNLS system [with two parameters δ and ϵ]

$$\begin{cases} i\,\tilde{q}_t + \tilde{q}_{xx} + i(4\delta - \epsilon)\tilde{q}\,\tilde{q}_x\tilde{r} + 4i\delta\tilde{q}^2\tilde{r}_x + \delta(4\delta + \epsilon)\,\tilde{q}^3\tilde{r}^2 = 0, \\ i\,\tilde{r}_t - \tilde{r}_{xx} + i(4\delta - \epsilon)\tilde{q}\,\tilde{r}\,\tilde{r}_x + 4i\delta\tilde{q}_x\tilde{r}^2 - \delta(4\delta + \epsilon)\,\tilde{q}^2\tilde{r}^3 = 0. \end{cases}$$

- DNLS system [with two parameters $(a-b)$ and κ]

$$\begin{cases} i\,\tilde{q}_t + \tilde{q}_{xx} + i\kappa(a-b-2)\tilde{q}\,\tilde{q}_x\tilde{r} + i\kappa(a-b-1)\tilde{q}^2\tilde{r}_x + \dfrac{\kappa^2(a-b)(a-b-1)}{4}\tilde{q}^3\tilde{r}^2 = 0, \\ i\,\tilde{r}_t - \tilde{r}_{xx} + i\kappa(a-b-2)\tilde{q}\,\tilde{r}\,\tilde{r}_x + i\kappa(a-b-1)\tilde{q}_x\tilde{r}^2 - \dfrac{\kappa^2(a-b)(a-b-1)}{4}\,\tilde{q}^2\tilde{r}^3 = 0, \end{cases}$$

where

$$\delta = \frac{\kappa(a-b-1)}{4}, \quad \epsilon = \kappa,$$

$$(a,b,\kappa) = \begin{cases} (0,0,1), & \text{(DNLS I, Kaup–Newell system)}, \\ (1,0,1), & \text{(DNLS II, Chen–Lee–Liu system)}, \\ (1,-1,1), & \text{(DNLS III, Ivanov–Gerdjikov system)}. \end{cases}$$

The general DNLS system

- corresponding linear system for the general DNLS (three complex parameters a, b, κ)

$$\frac{d}{dx}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}-i\lambda + (ib/2)\,\tilde{q}\,\tilde{r} & \kappa\,\sqrt{\lambda}\,\tilde{q}\\ (1/\kappa)\,\sqrt{\lambda}\,\tilde{r} & i\lambda + (ia/2)\,\tilde{q}\,\tilde{r}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \qquad x, t \in \mathbb{R}.$$

The transformation to the general DNLS system

- use the Kaup-Newell system as the unperturbed system
- view the general DNLS system as the perturbed system
- transform from the unperturbed to the perturbed system
- transform from $(a, b, \kappa) = (0, 0, 1)$ to the arbitrary set of parameters (a, b, κ)

$$q(x,t) \mapsto \tilde{q}(x,t) = \frac{1}{\kappa}\, q(x,t)\, E(x,t)^{b-a},$$

$$r(x,t) \mapsto \tilde{r}(x,t) = \kappa\, r(x,t)\, E(x,t)^{a-b},$$

$$E(x,t) := \exp\left(\frac{i}{2}\int_{-\infty}^{x} dz\, q(z,t)\, r(z,t)\right).$$

The DNLS I system (Kaup–Newell)

- Kaup–Newell system

$$\begin{cases} iq_t + q_{xx} - i(q^2 r)_x = 0, \\ ir_t - r_{xx} - i(q\, r^2)_x = 0, \end{cases} \qquad x, t \in \mathbb{R}.$$

- linear system associated with the Kaup–Newell system

$$\frac{d}{dx}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\lambda & \sqrt{\lambda}\, q \\ \sqrt{\lambda}\, r & i\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad x, t \in \mathbb{R}.$$

- NLS system

$$\begin{cases} iu_t + u_{xx} - 2u^2 v = 0, \\ iv_t - v_{xx} + 2u\, v^2 = 0, \end{cases} \qquad x, t \in \mathbb{R}.$$

- linear (AKNS) system associated with the NLS system

$$\frac{d}{dx}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -i\lambda & u \\ v & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad x, t \in \mathbb{R}.$$

The Marchenko method to solve integrable systems

- extend the Inverse Scattering Transform method

$$
\begin{array}{ccccc}
V(x,0) & \xrightarrow{\text{direct scattering at } t=0} & S(k,0) & \xrightarrow{\text{Fourier transform at } t=0} & F(y,0) \\
\text{solution} \downarrow & & \downarrow \text{time evolution} & & \downarrow \text{time evolution} \\
V(x,t) & \xleftarrow[\text{inverse scattering at time } t]{} & S(k,t) & \xleftarrow[\text{inverse Fourier transform at time } t]{} & F(y,t)
\end{array}
$$

- recover the potential from the solution to the Marchenko system

$$
\begin{array}{ccccc}
V(x,0) & \xrightarrow{\text{Marchenko kernel at } t=0} & F(y,0) & \xrightarrow{\text{time evolution}} & F(y,t) \\
\text{solution} \downarrow & & & & \downarrow \text{input to Marchenko system} \\
V(x,t) & \xleftarrow[\text{recover the potential}]{} & K(x,x^+,t) & \xleftarrow[\text{recover the potential}]{} & K(x,y,t)
\end{array}
$$

The Marchenko method for the Kaup–Newell system

- linear system of Marchenko integral equations

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y,t) & K_1(x,y,t) \\ \bar{K}_2(x,y,t) & K_2(x,y,t) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y,t) \\ \Omega(x+y,t) & 0 \end{bmatrix}$$

$$+ \int_x^\infty dz \begin{bmatrix} -iK_1(x,z,t)\,\Omega'(z+y,t) & \bar{K}_1(x,z,t)\,\bar{\Omega}(z+y,t) \\ K_2(x,z,t)\,\Omega(z+y,t) & i\bar{K}_2(x,z,t)\,\bar{\Omega}'(z+y,t) \end{bmatrix}, \qquad x < y.$$

- reflection coefficients $R(\sqrt{\lambda},0)$ and $\bar{R}(\sqrt{\lambda},0)$
- bound-state data via the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$

$$\begin{cases} \Omega(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \frac{R(\sqrt{\lambda},0)}{\sqrt{\lambda}}\, e^{4i\lambda^2 t} e^{i\lambda y} + C\, e^{4iA^2 t} e^{iAy}\, B, \\ \bar{\Omega}(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \frac{\bar{R}(\sqrt{\lambda},0)}{\sqrt{\lambda}}\, e^{-4i\lambda^2 t} e^{-i\lambda y} + \bar{C}\, e^{-4i\bar{A}^2 t} e^{-i\bar{A}y}\, \bar{B}. \end{cases}$$

The Marchenko method for the NLS system

- linear system of Marchenko integral equations

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y,t) & K_1(x,y,t) \\ \bar{K}_2(x,y,t) & K_2(x,y,t) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y,t) \\ \Omega(x+y,t) & 0 \end{bmatrix}$$

$$+ \int_x^\infty dz \begin{bmatrix} K_1(x,z,t)\,\Omega(z+y,t) & \bar{K}_1(x,z,t)\,\bar{\Omega}(z+y,t) \\ K_2(x,z,t)\,\Omega(z+y,t) & \bar{K}_2(x,z,t)\,\bar{\Omega}(z+y,t) \end{bmatrix}, \qquad x < y.$$

- reflection coefficients $R(\lambda, 0)$ and $\bar{R}(\lambda, 0)$
- bound-state data via the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$
- use of (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ allows any number of bound states with any multiplicities

$$\begin{cases} \Omega(y,t) := \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} d\lambda\, R(\lambda,0)\, e^{4i\lambda^2 t} e^{i\lambda y} + C\, e^{4iA^2 t} e^{iAy}\, B, \\ \bar{\Omega}(y,t) := \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} d\lambda\, \bar{R}(\lambda,0)\, e^{-4i\lambda^2 t} e^{-i\lambda y} + \bar{C}\, e^{-4i\bar{A}^2 t} e^{-i\bar{A}y}\, \bar{B}. \end{cases}$$

Contrast with recovery via the Marchenko method for the DNLS system

- recovery of potentials $q(x,t)$ and $r(x,t)$ and the key quantity $E(x,t)$

$$\begin{cases} q(x,t) = -2K_1(x,x,t)\exp\left(-4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right), \\ r(x,t) = -2\bar{K}_2(x,x,t)\exp\left(4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right), \\ E(x,t) = \exp\left(2\int_{-\infty}^x dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right). \end{cases}$$

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
- closed-form, compact formulas for explicit solutions involving matrix exponentials
- “unpacking” matrix exponentials yields explicit solutions in terms of elementary functions
- animation of explicit solutions via Mathematica

The reductions $r = \pm q^*$ in the Marchenko method for the Kaup–Newell

- reduced Marchenko equation in one dependent variable $K_1(x, y, t)$

$$K_1(x,y,t) \pm \Omega(x+y,t) \pm i \int_x^\infty dz\, K_1(x,z,t)\, \Omega'(z+s,t)\, \Omega(s+y,t)^* = 0, \qquad y > x.$$

- input $\Omega(y, t)$ to the Marchenko equation

$$\Omega(y,t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \frac{R(\sqrt{\lambda}, 0)}{\sqrt{\lambda}}\, e^{4i\lambda^2 t} e^{i\lambda y} + C\, e^{4iA^2 t} e^{iAy} B.$$

- recovery of the potential $q(x, t)$

$$q(x,t) = -2K_1(x,x,t)\, \exp\left(\mp 4i \int_x^\infty dz\, |K_1(z,z,t)|^2 \right).$$

- reflectionless case: input to the Marchenko equation

$$\Omega(y,t) = C\, e^{4iA^2 t} e^{iAy} B.$$

- kernel $\Omega(z + y, t)$ separable in z and y in the reflectionless case

$$\Omega(z+y,t) = C\, e^{4iA^2 t} e^{iAz} e^{iAy} B.$$

- closed-form, compact formulas for explicit solutions, animations via Mathematica

Reflectionless case: Explicit solutions for the Kaup–Newell system

- use (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as input to the Marchenko system
- obtain the Marchenko solution $K_1(x, y, t)$, $K_2(x, y, t)$, $\bar{K}_1(x, y, t)$, $\bar{K}_2(x, y, t)$.
- evaluate $K_1(x, x, t)$, $K_2(x, x, t)$, $\bar{K}_1(x, x, t)$, $\bar{K}_2(x, x, t)$
- obtain $q(x, t)$ and $r(x, t)$ via

$$\begin{cases} q(x,t) = -2K_1(x,x,t)\exp\left(-4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right), \\ r(x,t) = -2\bar{K}_2(x,x,t)\exp\left(4\int_x^\infty dz\,\left[\bar{K}_1(x,x,t) - K_2(x,x,t)\right]\right). \end{cases}$$

- animation of $|q(x, t)|$ and $|r(x, t)|$ using Mathematica
- Mathematica notebook using (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as input

Reflectionless case: Solution steps for the Kaup–Newell system

- construct M and $\bar{M}$ by solving

$$AM - M\bar{A} = iB\bar{C}, \qquad \bar{M}A - \bar{A}\bar{M} = i\bar{B}C.$$

- construct $\Gamma(x,t)$ and $\bar{\Gamma}(x,t)$ via

$$\begin{cases} \Gamma(x,t) := I - e^{iAx+4iA^2t}M\bar{A}e^{-2i\bar{A}x-4i\bar{A}^2t}\bar{M}e^{iAx}, \\ \bar{\Gamma}(x,t) := I - e^{-i\bar{A}x-4i\bar{A}^2t}\bar{M}Ae^{2iAx+4iA^2t}Me^{-i\bar{A}x}. \end{cases}$$

- construct $K_1(x,y,t)$, $K_2(x,y,t)$, $\bar{K}_1(x,y,t)$, and $\bar{K}_2(x,y,t)$ via

$$\begin{cases} K_1(x,y,t) = -\bar{C}\,e^{-i\bar{A}x}\,\bar{\Gamma}(x,t)^{-1}\,e^{-i\bar{A}y-4i\bar{A}^2t}\,\bar{B}, \\ K_2(x,y,t) = C\,e^{iAx}\,\Gamma(x,t)^{-1}\,e^{iAx+4iA^2t}M\bar{A}e^{-i\bar{A}(x+y)-4i\bar{A}^2t}\,\bar{B}, \\ \bar{K}_1(x,y,t) = \bar{C}\,e^{-i\bar{A}x}\,\bar{\Gamma}(x,t)^{-1}\,e^{-i\bar{A}x-4i\bar{A}^2t}\bar{M}A\,e^{iA(x+y)+4iA^2t}\,B, \\ \bar{K}_2(x,y,t) = -C\,e^{iAx}\,\Gamma(x,t)^{-1}\,e^{iAy+4iA^2t}\,B. \end{cases}$$

- obtain $q(x,t)$ and $r(x,t)$ explicitly in terms of (A,B,C) and $(\bar{A},\bar{B},\bar{C})$

Contrast with recovery via the Marchenko method for the NLS system

- recovery of potentials $u(x,t)$ and $v(x,t)$

$$\begin{cases} u(x,t) = -2K_1(x,x,t), \\ v(x,t) = -2\bar{K}_2(x,x,t). \end{cases}$$

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
- closed-form, compact formulas for explicit solutions involving matrix exponentials
- “unpacking” matrix exponentials yields explicit solutions in terms of elementary functions
- animation of explicit solutions via Mathematica

Example of an explicit solution to the DNLS system (Kaup–Newell)

- choose (a, b, κ) and the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as

$$a = 0, \quad b = 0, \quad \kappa = 1,$$

$$A = \left[5i\right], \quad B = \left[1\right], \quad C = \left[2\right], \quad \bar{A} = \left[-3i\right], \quad \bar{B} = \left[1\right], \quad \bar{C} = \left[3\right].$$

- we then obtain the corresponding potentials $q(x, t)$ and $r(x, t)$ as

$$q(x,t) = \frac{192\, e^{10(x+10it)} \left(32\, e^{16(x+4it)} - 15i\right)}{\left(32\, e^{16(x+4it)} + 9i\right)^2},$$

$$r(x,t) = \frac{128\, e^{6(x-6it)} \left(32\, e^{16(x+4it)} + 9i\right)}{\left(32\, e^{16(x+4it)} - 15i\right)^2}.$$

- we also obtain the key scalar quantity $E(x, t)$ as

$$E(x,t) = \frac{-96e^{16(x+4it)} + 45i}{160e^{16(x+4it)} + 45i}.$$

The snapshots for $|q(x,t)|$

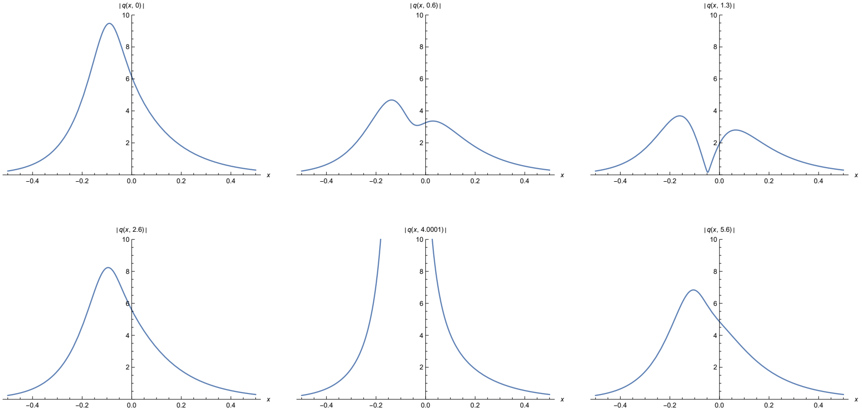

Figure: The snapshots for $|q(x,t)|$ at several t-values

Another example of an explicit solution to DNLS system (Kaup–Newell)

- choose (a, b, κ) and the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as

$$a = 0, \quad b = 0, \quad \kappa = 1,$$

$$A = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3i & 2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 3i \end{bmatrix}.$$

- we then obtain the corresponding potentials $q(x, t)$ and $r(x, t)$ as

$$q(x,t) = \frac{w_1 + w_2}{w_3^2}, \quad r(x,t) = \frac{w_4 + w_5}{w_6^2},$$

where

$$w_1 := -32e^{-6x+4it}\left[24t + 4e^{4x}(-3i + 16t + 4ix) - i(1 + 6x)\right],$$

$$w_2 := 9 + 16e^{4x}\left[5 - 14x + 24x^2 + 4e^{4x} + 8t(i + 48t)\right],$$

$$w_3 := -32\left[-7x + 84x^2 + 4t(48t - 11i)\right] + 73\cosh(4x) + 55\sinh(4x),$$

$$w_4 := 8e^{-6x-4it}\left[9(1 - 4x + 16i\,t) - 32e^{4x}(-1 + 3x + 12it)\right],$$

$$w_5 := 9 + 32e^{4x}\left[7x - 12x^2 + 2e^{4x} + 4t(11i - 48t)\right],$$

$$w_6 := 16\left[5 - 14x + 84x^2 + 8t(48t + i)\right] + 73\cosh(4x) + 55\sinh(4x).$$

Another example of an explicit solution to DNLS system (Kaup–Newell)

- we also obtain the key scalar quantity $E(x,t)$ as

$$E(x,t) = \left(\frac{w_7 + 128e^{8x} w_8}{w_9 + 128e^{8x} w_{10}} \right)^{1/2} \exp\left(i \tan^{-1}(128t\, e^{4x}/w_{11}) - i \tan^{-1}(1408t\, e^{4x}/w_{12}) \right),$$

where

$$w_7 := 81 + 4096e^{16x} + \left(288e^{4x} + 2048e^{12x}\right)\left(5 - 14x + 24x^2 + 384t^2\right),$$

$$w_8 := 59 - 280x + 872x^2 - 1344x^3 + 1152x^4 + 128t^2(61 - 168x + 288x^2) + 294912t^4,$$

$$w_9 := 81 + 4096e^{16x} - \left(576e^{4x} + 4096e^{12x}\right)\left(-7x + 12x^2 + 192t^2\right),$$

$$w_{10} := 9 + 392x^2 - 1344x^3 + 1152x^4 + 128t^2(121 - 168x + 288x^2) + 294912t^4,$$

$$w_{11} := 9 + 64e^{8x} + 16e^{4x}\left(5 - 14x + 24x^2 + 384t^2\right),$$

$$w_{12} := 9 + 64e^{8x} - 32e^{4x}\left(-7x + 12x^2 + 192t^2\right).$$

The snapshots for $|q(x,t)|$

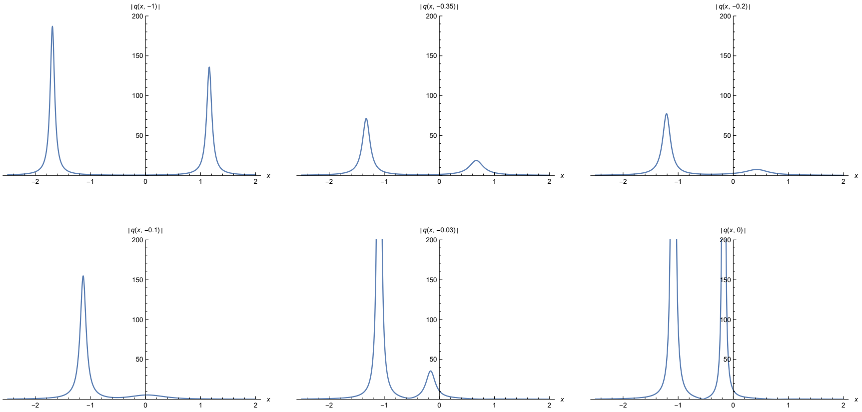

Figure: The snapshots for $|q(x,t)|$ at several t-values

The snapshots for $|r(x,t)|$

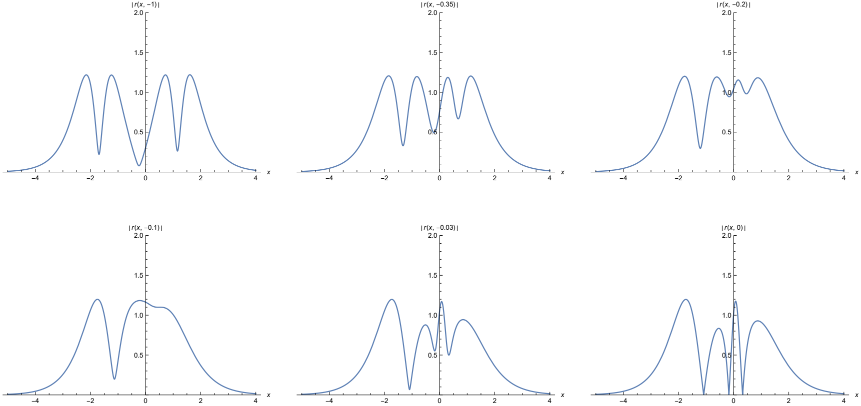

Figure: The snapshots for $|r(x,t)|$ at several t-values

Example of an explicit solution to the Chen–Lee–Liu system

- choose (a, b, κ) and the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as

$$a = 1, \quad b = 0, \quad \kappa = 1,$$

$$A = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} i & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -i & 1 \end{bmatrix}.$$

- we then obtain the corresponding potentials $\tilde{q}(x,t)$ and $\tilde{r}(x,t)$ as

$$\tilde{q}(x,t) = \frac{64\, w_{13}}{w_{14}} \exp\left(2x + 4it - 2i \tan^{-1}(w_{15}/w_{16})\right), \quad \tilde{r}(x,t) = \tilde{q}(x,t)^*,$$

where

$$w_{13} := x + 4it - 8\, e^{4x}(-i + 8t + 2ix),$$

$$w_{14} := -i + 256ie^{8x} + 32e^{4x}\left(1 - 4x + 8x^2 + 16it + 128t^2\right),$$

$$w_{15} := 32e^{4x}\left(1 - 4x + 8x^2 + 128t^2\right), \quad w_{16} := -1 + 256e^{8x} + 512e^{4x}t.$$

Example of an explicit solution to the Gerdjikov–Ivanov system

- choose (a, b, κ) and the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as

$$a = 1, \quad b = -1, \quad \kappa = 1,$$

$$A = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} i & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -i & 1 \end{bmatrix}.$$

- we then obtain the corresponding potentials $\tilde{q}(x, t)$ and $\tilde{r}(x, t)$ as

$$\tilde{q}(x, t) = \frac{64\, w_{17}}{w_{18}} \exp\left(2x + 4it\right), \quad \tilde{r}(x, t) = \tilde{q}(x, t)^*,$$

where

$$w_{17} := x + 4it - 8\, e^{4x}(-i + 8t + 2ix),$$

$$w_{18} := -i + 256ie^{8x} + 32e^{4x}\left(1 - 4x + 8x^2 + 16it + 128t^2\right).$$

Contrast with the paper by Kaup and Newell

- The Marchenko system in the paper by Kaup and Newell is not the same as our Marchenko system.
- Kaup and Newell studied only the reduced case.
- In our method, the Marchenko kernel is defined for any number of bound states with any multiplicities whereas in the paper by Kaup and Newell, it is defined only in the case where the bound states are simple.
- They came up with a very complicated Marchenko integral equation without satisfying any symmetries, whereas we have a very simple Marchenko integral equation which satisfies the symmetries.

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