On multiple completeness of root functions of certain classes of ordinary differential pencils of operators

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AMP Conference 2024
Analysis and Mathematical Physics
Mexico, August 5–17, 2024

August 14, 2024

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1. Problem statement, background and definitions

The main object considered in this report is an ordinary differential polynomial pencil $L(\lambda)$ in the space $L_2[0,1]$ generated by the differential expression (d.e.):

$$\ell(y,\lambda) := \sum_{j+s=n} p_{js}\lambda^s y^{(j)}, \quad p_{js} \in \mathbb{C},\ p_{n0} \neq 0,\ p_{0n} \neq 0, \tag{1}$$

and linearly independent boundary conditions:

$$U_i(y,\lambda) := \sum_{j=0}^{n-1} \alpha_{ij}(\lambda) y^{(j)}(0) + \beta_{ij}(\lambda) y^{(j)}(1) = 0, \quad i = \overline{1,n}, \tag{2}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, $\alpha_{ij}(\lambda)$, $\beta_{ij}(\lambda)$ are arbitrary polynomials in λ with complex coefficients.

We can assume that the boundary conditions (2) are normalized, that is, have the form

$$U_i(y,\lambda) \equiv U_{i0}(y,\lambda) + U_{i1}(y,\lambda) :=$$
$$:= \sum_{j+s\leqslant \varkappa_i} \lambda^s (\alpha_{ijs} y^{(j)}(0) + \beta_{ijs} y^{(j)}(1)) = 0, \quad i = \overline{1,n}, \tag{3}$$

α_{ijs}, $\beta_{ijs} \in \mathbb{C}$, $\varkappa_i$ is the order of the i-th boundary condition ($\varkappa_i \in \{0\} \cup \mathbb{N}$).

The total order of the boundary conditions (3) is denoted by the letter $\varkappa$, that is
$$\varkappa := \varkappa_1 + \varkappa_2 + \ldots + \varkappa_n. \tag{4}$$

The problem is to investigate n- and m-fold completeness in the space $L_2[0,1]$, where $1 \leqslant m \leqslant n$, the system of eigen- and associated functions (EAF) or, in a different way, of root functions (RF) of the pencil $L(\lambda)$.

The d.e. $\ell(y,\lambda)$ is supposed homogeneous, that is $p_{js} = 0$, $j+s<n$.

The boundary conditions $U_i(y,\lambda) = 0$ or homogeneous that is $\alpha_{ijs} = \beta_{ijs} = 0$, where $j+s<\varkappa_i$, or non-homogeneous.

Let the set of eigenvalues (EV) be $\Lambda := \{\lambda_k\}$ (it is supposed that it is countable). Let the corresponding system of the RF be $Y := \{y_k(x)\}$.

Definition 1

The system Y of root functions of the pencil $L(\lambda)$ is called *m-fold complete in the space* $L_2[0,1]$, where $1 \leqslant m \leqslant n$, if from the condition of orthogonality of the vector function (v.f.) $h \in L_2^m[0,1]$ to all derived (according to Keldysh) m-chains, corresponding to the system Y, the equality $h(x) = 0$ for a.e. $x \in [0,1]$ follows. Here $L_2^m[0,1]$ is defined by the formula

$$L_2^m[0,1] := \underbrace{L_2[0,1] \oplus \cdots \oplus L_2[0,1]}_{m \text{ times}}. \tag{5}$$

The fundamental work on this problem is the work of M. V. Keldysh

- Keldysh M. V. // Reports of the USSR Academy of Sciences. 1951. Vol. 77, No. 1. Pp. 11–14,

in which a theorem on the n-fold RF completeness was formulated without proof for the pencil $L(\lambda)$ generated by a d.e. with a special principal part, with variable coefficients and λ-independent splitting boundary conditions (when some of the boundary conditions are taken only at the end 0 of the segment $[0,1]$, and the rest at 1).

From the results of A. P. Khromov in his doctoral dissertation (1973) it follows that this theorem is valid in the case of analytic coefficients of the d.e.

Independently, this result in the case of analytical coefficients of the d.e. was obtained by W. Eberhard (1976).

A. A. Shkalikov (1976) established this result in the case of summable coefficients of the d.e.

A generalization of this theorem to the case of a finite-dimensional perturbation of a Volterra operator was made by A. P. Khromov (1977).

The case of an arbitrary principal part of a differential equation was considered by G. Freiling (1984) and S. A. Tikhomirov (1987).

In the works of M. G. Gasymov and A. M. Magerramov (1974), as well as A. A. Shkalikov (1983), related to the general form of the pencil $L(\lambda)$, sufficient conditions for n-fold RF completeness in $L_2[0,1]$ are obtained in terms of the power boundedness in the parameter λ of the Green's function of the pencil $L(\lambda)$ on some rays in the complex plane.

The most complete study of the question of n- and m-fold RF completeness for the pencils with constant coefficients of the d.e., whose boundary conditions are semi-splitting ($l \geqslant n-l$) and independent of λ, was carried out by A. I. Vagabov (in the 80th years)

The author managed to obtain (see the papers [1-4]) new results in solution of this problem in the case of the pencil $L(\lambda)$ with constant coefficients of it's d.e. and boundary conditions as splitting and also polynomial depending from spectral parameter λ, and some classes of non-splitting boundary conditions.

Consider the equation

$$\ell(y,\lambda)=0. \tag{6}$$

Assume that the roots $\{\omega_k\}_{k=1}^{n}$ of its characteristic equation (we call them characteristics for short)

$$\sum_{j+s=n} p_{js}\omega^j = 0$$

are pairwise distinct and nonzero. The system of functions

$$y_j(x,\lambda)=e^{\lambda\omega_j x}, \quad j=\overline{1,n}, \tag{7}$$

is a fundamental system of solutions (f.s.s.) of the equation $\ell(y,\lambda)=0$ for $\lambda\neq 0$.

2. Classification of the pencils $L(\lambda)$ (according to A.A. Shkalikov)

When studying the multiple RF completeness for the pencil $L(\lambda)$, an important role is played by the class to which $L(\lambda)$ belongs (the class of regular pencils or irregular ones). The methods for studying these pencils depend significantly on this. Therefore, it is necessary to first classify pencils. It is very convenient to use the approach proposed by Shkalikov (1983). In the case of d.e. (1), the f.s.s. has the simplest form (7) and the classification is greatly simplified and becomes finite, unlike Shkalikov's classification in the general case.

Let us introduce the following column vectors for $j = \overline{1, n}$

$$U_j(\lambda) \equiv (u_{1j}(\lambda), \dots, u_{nj}(\lambda))^T := (U_1(y_j, \lambda), \dots, U_n(y_j, \lambda))^T , \tag{8}$$

$$V_j(\lambda) \equiv (v_{1j}(\lambda), \dots, v_{nj}(\lambda))^T := (U_{10}(y_j, \lambda), \lambda), \dots, U_{n0}(y_j, \lambda))^T , \tag{9}$$

$$W_j(\lambda) \equiv (w_{1j}(\lambda), \dots, w_{nj}(\lambda))^T := e^{-\lambda \omega_j} (U_{11}(y_j, \lambda), \dots, U_{n1}(y_j, \lambda))^T . \tag{10}$$

For brevity, the λ argument will sometimes be omitted.

Let's introduce a set

$$\Omega = \{0, \omega_i, \omega_i + \omega_j (i \neq j), \omega_i + \omega_j + \omega_k (i \neq j \neq k), \dots, \omega_1 + \dots + \omega_n\}. \quad (11)$$

Then the characteristic determinant of the pencil will have the form

$$\Delta(\lambda) = \det \left(U_i(y_j, \lambda)\right)_{i,j=1}^{n} =$$
$$= \left|V_1(\lambda) + e^{\lambda \omega_1} W_1(\lambda), \dots, V_n(\lambda) + e^{\lambda \omega_n} W_n(\lambda)\right| = \lambda^{\varkappa} \sum_{\omega \in \Omega} F^{\omega}(\lambda) e^{\lambda \omega}, \quad (12)$$

where the coefficients $F^{\omega}(\lambda)$ are finite sums:

$$F^{\omega}(\lambda) = F_0^{\omega} + \frac{1}{\lambda} F_1^{\omega} + \dots + \frac{1}{\lambda^{\varkappa}} F_{\varkappa}^{\omega}, \quad (13)$$

here F_j^{ω} are determinants independent of λ.

It is known that non-zero eigenvalues of $L(\lambda)$ are zeros of $\Delta(\lambda)$. Specific values of eigenvalues or their asymptotics are not required below. We denote $\Lambda_0 := \Lambda \setminus \{0\}$

Further, the following notation will be used

$$[\eta(x,\lambda)]_r = \eta_0(x) + \frac{\eta_1(x)}{\lambda} + \ldots + \frac{\eta_r(x)}{\lambda^r}, \quad r \in \{0\} \cup \mathbb{N} \tag{14}$$

for the function

$$\eta(x,\lambda) = \eta_0(x) + \frac{\eta_1(x)}{\lambda} + \ldots + \frac{\eta_r(x)}{\lambda^r} + \frac{\eta_r(x)}{\lambda^{r+1}} + \ldots . \tag{15}$$

For $r \in \{0, 1, \ldots, \varkappa\}$, by $(M_\Delta)_r$ we denote the convex hull of those points ω for which $[\mathrm{F}^\omega(\lambda)]_r \not\equiv 0$. It is clear that

$$(M_\Delta)_0 \subset (M_\Delta)_1 \subset \ldots \subset (M_\Delta)_\varkappa \subset M. \tag{16}$$

We briefly denote the polygon $(M_\Delta)_\varkappa$ as M_Δ and call it the *characteristic polygon* (c.p.) of the function $\Delta(\lambda)$. It should be noted that M_Δ is exactly the conjugate diagram (c.d.) $\bar{I}_\Delta$ of an entire function of finite degree (or exponential type) $\Delta(\lambda)$, represented as (12), in accordance with the definition from the book

- Levin B. Ya. Distribution of zeros of entire functions. Vol. 5 Providence: AMS, 1980.

Similar to Shkalikov, we give the following definitions.

Definition 2

A pencil $L(\lambda)$ is called *regular* (regular in the sense of Birkhoff-Tamarkin) if $(M_\Delta)_0 = M$.

A. A. Shkalikov (1983) noted that the definition of regularity 1 of the pencil $L(\lambda)$ is equivalent to the definition of regularity according to J. D. Tamarkin (1917) and, naturally, such regular pencils are called regular in the sense of Tamarkin. In the special case of ordinary differential operators, this regularity coincides with the regularity according to Birkhoff.

Definition 3

The pencil $L(\lambda)$ is called *almost regular* (regular in the sense of Stone) if $(M_\Delta)_\varkappa = M$.

The pencils that are called almost regular in the definition 1 are a generalization of one class of ordinary differential operators that were studied for order $n = 2$ by M. Stone (1927), and in the general case by A. P. Khromov (1962) and G. E. Benzinger (1970, 1972) (G. E. Benzinger called such operators regular in the sense of Stone).

Definition 4

The pencil $L(\lambda)$ is called *weakly irregular* (or *normal* in the terminology of Shkalikov) if the polygon $\left(M_\Delta\right)_\varkappa$ has at least two points of tangency with M, and the perpendiculars drawn from some fixed interior point to the sides of M on which the points of tangency lie (if the point of tangency is angular, then there are two such perpendiculars) divide the complex plane into sectors of angle $< \pi$.

That is, the pencil $L(\lambda)$ is weakly irregular if there are at least three rays in the λ-plane that split the complex plane into sectors of angle $< \pi$ and on which the resolvent of the pencil $L(\lambda)$ has at most a power growth.

The class of weakly irregular pencils $L(\lambda)$ given by the definition of 1 was introduced by A. A. Shkalikov (1983) and was called the class of normal pencils. This class is significantly wider than the first two classes. It follows from the results of A. A. Shkalikov that if the pencil $L(\lambda)$ is from this class, then the RF system is n-fold complete in $L_2[0,1]$.

Definition 5

A pencil $L(\lambda)$ that does not satisfy the previous definition is called *strongly irregular*.

In the figure 1 the polygon M is indicated by the black line, and the polygon M_Δ is indicated by the red line.

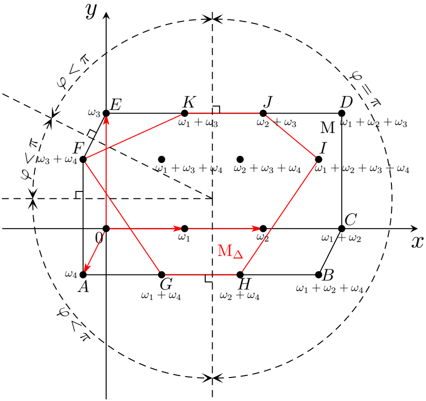

Fig. 1: Polygons of M and M_Δ. Strongly irregular pencil

That is, the pencil $L(\lambda)$ is strongly irregular if its resolvent has an exponential growth in the λ-plane no less than in the half-plane.

The class of strongly irregular pencils includes pencils whose resolvents have exponential growth in sectors of the complex plane of angle $\geqslant \pi$. The study of multiple completeness the RF system for pencils from this class presents the greatest difficulty and has been carried out only for individual sets of pencils $L(\lambda)$.

3. Generating functions and the scheme of proving of multiple completeness

The main concept below is the concept of a generating function.

Definition 6

The function $g(x,\lambda)$, defined for all $x \in [0,1]$ and $\lambda \in \mathbb{C}$, will be called the *generating function* (g.f.) for the RF system of the pencil $L(\lambda)$, if the functions

$$\frac{1}{q!}\frac{\partial^q g(x,\lambda)}{\partial \lambda^q}\bigg|_{\lambda=\lambda_\nu}, \quad q=\overline{0,s_\nu}, \quad \lambda_\nu \in \Lambda, \tag{17}$$

are the RF of the pencil $L(\lambda)$ corresponding to the eigenvalue λ_ν of multiplicity $s_\nu + 1$.

In the case of simple eigenvalues of the pencil $L(\lambda)$ the function $g(x,\lambda)$ will be generating for the system of eigenfunctions (EF) if the system $\{g(x,\lambda_\nu)\}_{\lambda_\nu \in \Lambda}$ is the EF system of $L(\lambda)$.

The solutions of the equation $\ell(y,\lambda) = 0$ are usually taken as g.f.'s, i.e. the g.f. is a combination of exponentials of the form (7).

Therefore, the g.f.'s are entire functions of finite degree in λ. Then for each fixed $x \in [0,1]$ we can construct the conjugate diagram

$$M_{\mathrm{g}(x,\lambda)} := \bar{I}_{\mathrm{g}(x,\lambda)}, \tag{18}$$

which is a polygon.

We introduce the set

$$M_{\mathrm{g}(\cdot,\lambda)} := \mathrm{conv}_{x\in[0,1]} M_{\mathrm{g}(x,\lambda)}. \tag{19}$$

Since $g(x,\lambda)$ is a combination of exponentials (7), a simpler formula is valid

$$M_{\mathrm{g}(\cdot,\lambda)} = \mathrm{conv}\{M_{\mathrm{g}(0,\lambda)}, M_{\mathrm{g}(1,\lambda)}\}. \tag{20}$$

We will call $M_{\mathrm{g}(\cdot,\lambda)}$ the characteristic polygon (ch.p.) of the function $\mathrm{g}(x,\lambda)$

From the point of view of the scheme of proof multiple RF completenes is preferable to the g.f.'s with the most compact ch.p. in the following sense.

Definition 7

We will say that the g.f. $\mathrm{g}(x,\lambda)$ satisfies *the condition* (α) if M_Δ has at least two points of tangency with $M_{\mathrm{g}(\cdot,\lambda)}$, and the perpendiculars drawn from some fixed interior point to the sides of $M_{\mathrm{g}(\cdot,\lambda)}$ on which the points of tangency of M_Δ lie (if the point of tangency is angular, then there are two such perpendiculars) divide the complex plane into sectors of angel $< \pi$.

In the figure 2 the polygon M is indicated by the black line, the polygon M_Δ is indicated by the red line and the polygon $M_{g(\cdot,\lambda)}$ is indicated by the blue dashed line.

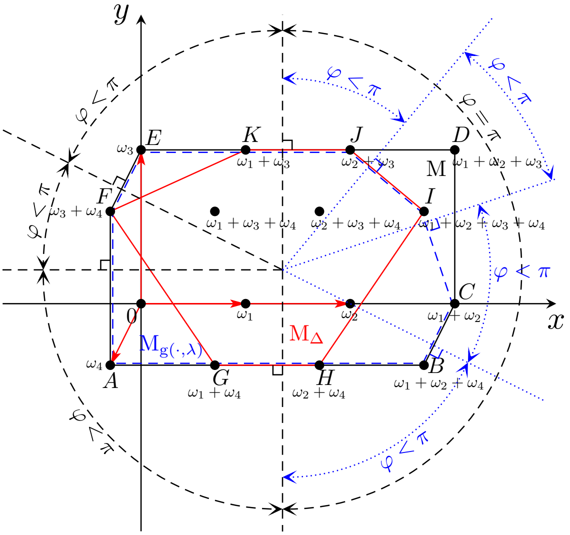

Fig. 2: Strongly irregular pencil, $g(x, \lambda) \in (\alpha)$

From the definition of g.f.'s one can directly verify that the vectors

$$\left(\frac{\partial^q \mathrm{g}(x,\lambda)}{\partial \lambda^q}, \frac{\partial^q (\lambda \mathrm{g}(x,\lambda))}{\partial \lambda^q}, \dots, \frac{\partial^q (\lambda^{m-1} \mathrm{g}(x,\lambda))}{\partial \lambda^q} \right)^T \Bigg|_{\lambda=\lambda_\nu}, \tag{21}$$

where $q = \overline{0, s_\nu}$, $\lambda_\nu \in \Lambda$, are the derivative m-chains for the root function of $L(\lambda)$, corresponding to the eigenvalue λ_ν of multiplicity $s_\nu + 1$.

The standard scheme of proving of multiple completeness (abbreviated as the PMC scheme) is based on the following lemma.

Lemma 1

If g.f. $\mathrm{g}(x,\lambda) \in (\alpha)$ *and* $h(x) = (h_1(x), h_2(x), \dots, h_m(x))^T$ *is orthogonal to all derivative* m*-chains corresponding to the RF system* Y*, then under the additional condition of orthogonality* $h(x)$ *to some finite set of v.f.'s the equality*

$$H_m(\mathrm{g},\lambda) \equiv 0 \tag{22}$$

holds, where

$$H_m(\mathrm{g},\lambda) := \int_0^1 \mathrm{g}(x,\lambda)\mathrm{h}_m(x,\lambda)\,dx, \quad \mathrm{h}_m(x,\lambda) := \sum_{j=1}^m \lambda^{j-1} \bar{h}_j(x) \tag{23}$$

Consider the meromorphic function

$$\mathcal{H}_m(\mathrm{g},\lambda) := \frac{H_m(\mathrm{g},\lambda)}{\Delta(\lambda)}. \tag{24}$$

This function formally has poles at the points $\lambda_\nu \in \Lambda$, but if the assumptions of the lemma 1 are satisfied, all the poles of this function are compensated by the numerator. That is, $\mathcal{H}_m(\mathrm{g},\lambda)$ is an entire function of finite degree.

The fulfillment of the condition (α) for the g.f. $\mathrm{g}(x,\lambda)$ means that the function $\mathcal{H}_m(\mathrm{g},\lambda)$ satisfies the Phragmén-Lindelöf principle.

The PMC scheme for the pencil $L(\lambda)$ assumes the existence of some set of g.f.'s $\mathrm{g}_i(x,\lambda)$, $i=\overline{1,r}$, satisfying condition (α). Then, based on lemma 1, we obtain

$$H_m(\mathrm{g}_i,\lambda) \equiv 0, \quad i=\overline{1,r}, \tag{25}$$

under the condition that v.f. $h(x)$ is orthogonal to all derivative m-chains corresponding to the RF system and, possibly, to some additional finite set of v.f.'s. If the set of g.f.'s is sufficient to obtain from (25) the relations

$$h_1(x) = h_2(x) = \ldots = h_m(x) = 0 \quad \text{ for a.e. } x \in [0,1], \tag{26}$$

then this means that there is m-fold RF completeness for the pencil $L(\lambda)$ in the space $L_2[0,1]$ with possible finite defect.

An essential condition for the described PMC scheme is the presence of a sufficient number of g.f.'s $\mathrm{g}_i(x) \in (\alpha)$.

When studying the spectral properties of differential pencils, g.f.'s of the form

$$g_j(x,\lambda) = \begin{vmatrix} U_1(y_1,\lambda) & U_1(y_2,\lambda) & \dots & U_1(y_n,\lambda) \\ \dots & \dots & \dots & \dots \\ U_{j-1}(y_1,\lambda) & U_{j-1}(y_2,\lambda) & \dots & U_{j-1}(y_n,\lambda) \\ y_1(x,\lambda) & y_2(x,\lambda) & \dots & y_n(x,\lambda) \\ U_{j+1}(y_1,\lambda) & U_{j+1}(y_2,\lambda) & \dots & U_{j+1}(y_n,\lambda) \\ \dots & \dots & \dots & \dots \\ U_n(y_1,\lambda) & U_n(y_2,\lambda) & \dots & U_n(y_n,\lambda) \end{vmatrix}, \quad j = \overline{1,n}, \tag{27}$$

are usually used (Naimark, Shkalikov, Vagabov, etc.).

These functions, which we will call *classical g.f.'s*, are entire functions in λ, linearly independent for $x \in [0,1]$ and $\lambda \notin \Lambda$.

Generating functions of the form (27) have been successfully used in the study of pencils $L(\lambda)$ with splitting or semi-splitting boundary conditions under some additional assumptions on the characteristics.

Carrying out the above-described scheme of proving the multiple RF completeness for the pencil $L(\lambda)$ becomes much more complicated or does not work at all if either $g_i(x), \lambda) \notin (\alpha)$ for all $i = \overline{1,n}$, i.e. the functions $\mathcal{H}_m(\mathrm{g}_i, \lambda)$ have exponential growth at least in the half-plane of the λ-plane, or it is problematic to derive equalities (26) from the set of relations (25). Following the described scheme, in these situations it is not possible to obtain m-multiple RF completeness in $L_2[0, 1]$.

To expand the capabilities of the described scheme for proving the multiple RF completeness, the author proposes in 1996–2015 (see details in the paper [1]) to use *generalized g.f.'s* of the form

$$g(x, \lambda, \Gamma(\lambda)) := \gamma_1(\lambda) g_1(x, \lambda) + \gamma_2(\lambda) g_2(x, \lambda) \cdots + \gamma_n(\lambda) g_n(x, \lambda), \tag{28}$$

where

$$\Gamma(\lambda) = \big(\gamma_1(\lambda), \gamma_2(\lambda), \ldots, \gamma_n(\lambda)\big)^T \not\equiv (0, 0, \ldots, 0)^T \tag{29}$$

is a vector parameter.

For $g(x, \lambda, \Gamma(\lambda))$ it is also convenient to use the following representation

$$g(x, \lambda, \Gamma(\lambda)) := \begin{vmatrix} 0 & y_1(x,\lambda) & \dots & y_n(x,\lambda) \\ -\Gamma(\lambda) & U_1(\lambda) & \dots & U_n(\lambda) \end{vmatrix}, \tag{30}$$

where the column vectors $U_j(\lambda)$ are defined by the formulas (8).

In particular, it follows from the formulas (27) and (30) that the k.p.f.'s is a special case of the generalized g.f.'s, namely:

$$g_i(x, \lambda) = g(x, \lambda, E_i), \quad i = \overline{1, n} \tag{31}$$

where $E_i = (\delta_{i1}, \dots, \delta_{in})^T$, $i = \overline{1, n}$, are unit vectors. Here δ_{ik} denotes the Kronecker delta.

The following lemma is true.

Lemma 2

For any $\lambda \in \mathbb{C} \setminus \Lambda_0$, the functions $g(x, \lambda, \Gamma_j(\lambda))$, $j = \overline{1, n}$, are linearly independent with respect to the variable $x \in [0, 1]$ if and only if the v.f.'s $\Gamma_j(\lambda)$, $j = \overline{1, n}$, with respect to λ, are linearly independent.

Next, by Ω_j we denote the subset of those points from Ω that can be represented as $\omega_j + \ldots$, i.e., contain the number ω_j as an addend. By Ω^j we denote the set $\Omega \setminus \Omega_j$, i.e., those points from Ω that do not contain the number ω_j as an addend.

Expanding the determinant (30) along the first line, we obtain

$$g(x,\lambda,\Gamma) = \sum_{k=1}^{n} y_k(x,\lambda)\Big|U_1,\ldots,U_{k-1},\Gamma,U_{k+1},\ldots,U_n\Big| = \lambda^{\varkappa}\sum_{k=1}^{n} e^{\lambda\omega_k x}\times$$

$$\times\Big|\hat{V}_1+e^{\lambda\omega_1}\hat{W}_1,\ldots,\hat{V}_{k-1}+e^{\lambda\omega_{k-1}}\hat{W}_{k-1},\hat{\Gamma},\hat{V}_{k+1}+e^{\lambda\omega_{k+1}}\hat{W}_{k+1},\ldots,\hat{V}_n+e^{\lambda\omega_1}\hat{W}_n\Big| =$$

$$= \lambda^{\varkappa}\sum_{k=1}^{n}\sum_{\omega\in\Omega^k} \mathrm{G}_k^{\omega}(\lambda)e^{\lambda(\omega_k x+\omega)}, \quad (32)$$

where the vectors with hats have the following form ($j=\overline{1,n}$)

$$\hat{V}_j(\lambda) = \left(\frac{1}{\lambda^{\varkappa_1}}v_{1j}(\lambda),\ldots,\frac{1}{\lambda^{\varkappa_n}}v_{nj}(\lambda)\right)^T, \ \hat{W}_j(\lambda) = \left(\frac{1}{\lambda^{\varkappa_1}}w_{1j}(\lambda),\ldots,\frac{1}{\lambda^{\varkappa_n}}w_{nj}(\lambda)\right)^T \quad (33)$$

It is convenient to assume that the vector $\Gamma(\lambda)$ is a vector polynomial in λ of the form $\Gamma(\lambda) = (\gamma_1(\lambda),\ldots,\gamma_n(\lambda))^T$, where

$$\gamma_j(\lambda) = \gamma_{j\varkappa_j}\lambda^{\varkappa_j} + \gamma_{j,\varkappa_j-1}\lambda^{\varkappa_j-1} + \ldots + \gamma_{j0}, \quad j=\overline{1,n}. \quad (34)$$

By construction, $\mathrm{G}_k^\omega(\lambda) = O(1)$ for $|\lambda| \gg 1$. From formula (32) it is clear that the generalized g.f. $g(x, \lambda, \Gamma(\lambda))$ is an entire function of finite degree in λ. For such a function, the ch.p. was already introduced. Since for a fixed pencil $L(\lambda)$ the form of this polygon is determined only by the vector-parameter $\Gamma(\lambda)$, we call it the ch.p. of the vector-parameter $\Gamma(\lambda)$ and denote it by $M(\Gamma(\lambda))$. That is, by definition

$$M(\Gamma(\lambda)) := M_{g(\cdot,\lambda,\Gamma(\lambda))}. \tag{35}$$

It is quite clear that $M_\Delta \subset M(\Gamma(\lambda)) \subset M$.

We will say that $\Gamma(\lambda)$ satisfies condition (α) and write $\Gamma(\lambda) \in (\alpha)$ if $g(x, \lambda, \Gamma(\lambda)) \in (\alpha)$.

Thus, from the PMC scheme it follows that if the vector-parameter $\Gamma(\lambda) \in (\alpha)$, then under the condition of orthogonality of the v.f. $h(x)$ to all derivative m-chains corresponding to the RF system and, possibly, to some additional finite set of v.f.'s from $L_2[0,1]$ we obtain by the lemma 1 the identity

$$H_m(\Gamma(\lambda), \lambda) \equiv 0. \tag{36}$$

It is clear from the above that when choosing the vector-parameter $\Gamma(\lambda)$, one should be guided by the fact that the polygon $M(\Gamma)$ is as "compact" as possible (as close as possible to M_Δ).

4. Sufficient conditions for multiple RF completeness using the generalized g.f.'s

An important role in constructing the generalized g.f.'s are played by the vector parameters $V_j(\lambda)$ and $W_j(\lambda)$. This is due to the fact that in the formula (32) for the generalized g.f. in the case of $\Gamma(\lambda) = V_j(\lambda)$ or $\Gamma(\lambda) = W_j(\lambda)$ a great many exponents vanish due to the corresponding coefficients vanishing due to the presence of identical columns in the determinants defining them.

When checking the condition (α) for the vector parameters $V_j(\lambda)$ and $W_j(\lambda)$, it is convenient to use the rather simple sufficient conditions.

Lemma 3

For a fixed index j $(1 \leqslant j \leqslant n)$, the inclusion $M(V_j) \subset \operatorname{conv}\{M_\Delta, \Omega_j\}$ holds.

Lemma 4

For a fixed index j $(1 \leqslant j \leqslant n)$, the inclusion $M(W_j) \subset \operatorname{conv}\{M_\Delta, \Omega^j\}$ holds.

By construction, the following inclusions are valid

$$M_\Delta \subset M(V_j) \subset \operatorname{conv}\{M_\Delta, \Omega_j\} \subset M, \;\; M_\Delta \subset M(W_j) \subset \operatorname{conv}\{M_\Delta, \Omega^j\} \subset M. \tag{37}$$

Let us formulate sufficient conditions, which are essentially the most important results of the report.

Theorem 1

If there exist n linearly independent vector parameters $\Gamma_1(\lambda)$, $\Gamma_2(\lambda)$, $\ldots$, $\Gamma_n(\lambda) \in (\alpha)$, then the RF system of the pencil $L(\lambda)$ is n-fold complete in $L_2[0,1]$ with zero defect in the case $\varkappa_i \leqslant n-1$, $i=\overline{1,n}$, and with possible finite defect otherwise.

Corollary 1

If $V_{i_s} \in (\alpha)$, $s=\overline{1,k}$, $W_{j_r} \in (\alpha)$, $r=\overline{1,l}$, $k+l \geqslant n$ and

$$\operatorname{rank}\left(V_{i_1}, V_{i_2}, \ldots, V_{i_k}, W_{j_1}, W_{j_2}, \ldots, W_{j_l}(\lambda)\right) = n, \tag{38}$$

then the RF system of the pencil $L(\lambda)$ is n-fold complete in $L_2[0,1]$.

Theorem 2

If there exist m pairs of vectors $\{V_{j_s}, W_{j_s}\}$, $s=\overline{1,m}$, such that V_{j_s}, $W_{j_s} \in (\alpha)$, then the RF system of the pencil $L(\lambda)$ is m-fold complete in $L_2[0,1]$ with possible finite defect.

5. The case of a pencil with characteristics on two rays and splitting boundary conditions

In the space $L_2[0,1]$ we consider an ordinary differential polynomial pencil $L^1(\lambda)$ generated by the d.e. $\ell(y,\lambda)$ of the form

$$\ell(y,\lambda) := \sum_{j+s=n} p_{js}\lambda^s y^{(j)},\ p_{js} \in \mathbb{C},\ p_{n0} \neq 0,\ p_{0n} \neq 0, \tag{39}$$

and linearly independent homogeneous two-point splitting normalized boundary conditions

$$U_i^1(y,\lambda) := \sum_{j+s=\varkappa_i} \lambda^s \alpha_{ijs} y^{(j)}(0) = 0, \quad i = \overline{1,l}, \tag{40}$$

$$U_i^1(y,\lambda) := \sum_{j+s=\varkappa_i} \lambda^s \beta_{ijs} y^{(j)}(1) = 0, \quad i = \overline{l+1,n}, \tag{41}$$

where

$$\lambda, \alpha_{ijs}, \beta_{ijs} \in \mathbb{C},\ \varkappa_i, \in \mathbb{N} \cup \{0\},\ 1 \leqslant l \leqslant n-1. \tag{42}$$

Let us assume that the characteristics $\omega_1, \omega_2, \ldots, \omega_n$ are pairwise distinct, nonzero and lie on two or one rays emanating from the origin, in quantities k and $n-k$ $(0 \leqslant k \leqslant n)$.

Without loss of generality, we can assume that

$$\omega_n e^{i(\pi-\varphi)} < \omega_{n-1} e^{i(\pi-\varphi)} < \cdots < \omega_{k+1} e^{i(\pi-\varphi)} < 0 < \omega_1 < \omega_2 < \cdots < \omega_k, \tag{43}$$

where $0 < |\varphi| < \pi$ (see figure 3). That is, the first k roots of ω_j lie on the positive ray, and the remaining $n-k$ roots lie on the ray emanating from the origin at an angle of φ.

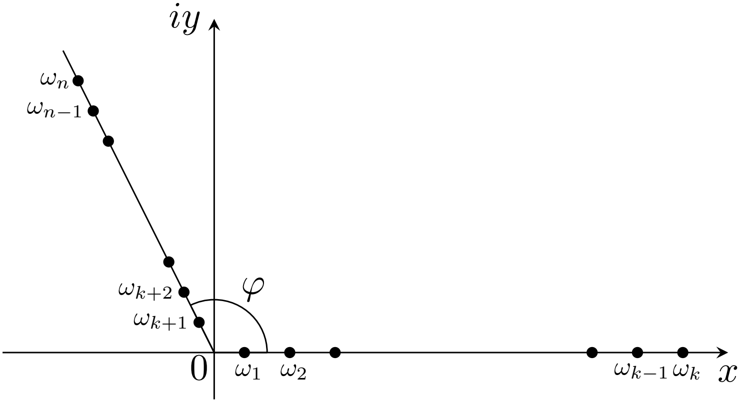

Fig. 3: Location of characteristics

To formulate the theorems, we introduce the following notation.

$$a_{ij} = \sum_{\nu+s=\varkappa_i} \alpha_{i\nu s}\omega_j^{\nu},\ i=\overline{1,l};\ b_{ij} = \sum_{\nu+s=\varkappa_i} \beta_{i\nu s}\omega_j^{\nu},\ i=\overline{l+1,n},\ j=\overline{1,n}. \quad (44)$$

$$\det(a_{ij})_{i=\overline{1,l}}^{j=\overline{1,k+l-n};\overline{k+1,n}} \neq 0, \quad \det(b_{ij})_{i=\overline{l+1,n}}^{j=\overline{k+l-n+1,k}} \neq 0 \quad \text{при } n-k \leqslant l; \quad (45)$$

$$\det(a_{ij})_{i=\overline{1,l}}^{j=\overline{n-l+1,n}} \neq 0, \quad \det(b_{ij})_{i=\overline{l+1,n}}^{j=\overline{1,n-l}} \neq 0 \quad \text{при } n-k \geqslant l; \quad (46)$$

$$\det(a_{ij})_{i=\overline{1,l}}^{j=\overline{1,l}} \neq 0, \quad \det(b_{ij})_{i=\overline{l+1,n}}^{j=\overline{l+1,n}} \neq 0 \quad \text{при } k \leqslant l; \quad (47)$$

$$\det(a_{ij})_{i=\overline{1,l}}^{j=\overline{k-l+1,k}} \neq 0, \quad \det(b_{ij})_{i=\overline{l+1,n}}^{j=\overline{1,k-l};\overline{k+1,n}} \neq 0 \quad \text{при } k \geqslant l. \quad (48)$$

Lemma 5

The classical g.f.'s $g_i(x,\lambda)$ *satisfy condition* (α) *if*

a) $\max\{k, n-k\} \leqslant l$, $i = \overline{l+1, n}$, *under the conditions* (45) *and* (47),

b) $\min\{k, n-k\} \geqslant l$, $i = \overline{1, l}$, *under the conditions* (46) *and* (48),

and at the same time the estimates are valid for $|\lambda| \gg 1$

$$|\mathcal{H}_m(g_i,\lambda)| \leqslant C(\varepsilon)|\lambda|^{m-\frac{3}{2}-\varkappa_i}. \qquad (49)$$

Remark 1

In the case

$$\min\{k, n-k\} < l < \max\{k, n-k\} \qquad (50)$$

it cannot be guaranteed that $g_i(x,\lambda) \in (\alpha)$ for some $i \in \{1, n\}$. That is, using the classical g.f.'s $g_i(x,\lambda)$, $i = \overline{1, n}$, it is currently not possible to establish the multiple RF completeness for the pencil $L^1(\lambda)$. This situation has not been previously identified in the study of the multiple RF completeness for the pencil $L^1(\lambda)$.

Theorem 3

If $\max\{k, n-k\} \leqslant l$ *and conditions* (45) *and* (47) *are satisfied, then for* $m = 2(n-l)$ *the RF system of the pencil* $L^1(\lambda)$ *is* m*-fold complete in the space* $L_2[0,1]$ *with possible finite defect.*

Theorema 4

If $\min\{k, n-k\} \geqslant l$ *and conditions* (46) *and* (48) *are satisfied, then for* $m = 2l$ *the RF system of the pencil* $L^1(\lambda)$ *is* m*-multiple complete in the space* $L_2[0,1]$ *with possible finite defect.*

In the case $[k, n-k]_- < l < [k, n-k]_+$, reasoning using only the classical g.f.'s does not work, since the classical g.f.'s do not satisfy, generally speaking, the condition (α).

It turns out that in the case of (50) one can construct generalized g.f.'s satisfying the (α) condition, thereby implementing the PMC scheme and obtaining conditions for the multiple RF completeness for the pencil $L^1(\lambda)$.

Let the characteristics of the pencil $L^1(\lambda)$ be located on two rays emanating from the origin, and let condition

$$\min\{k, n-k\} < l < \max\{k, n-k\} \tag{51}$$

be satisfied. Further for definiteness we consider that $n-k<k$.

In this case, the polygon M is a quadrangle $A'B'C'D'$, which is a parallelogram (figures 4 and 5, the figures are given for the case $n-k \leqslant k$).

In the case of $n-k \leqslant k$, we will conventionally call the sides $A'B'$ and $D'C'$ the "long" sides of the parallelogram $A'B'C'D'$ (in reality, these sides can actually be shorter or longer than the sides $A'D'$ and $B'C'$ — it all depends on the values of the characteristics lying on $A'B'$ and $A'D'$). The term "longer" means only that the sides $A'B'$ and $D'C'$ are generated by no fewer characteristics than the sides $A'D'$ and $B'C'$, due to the fact that $n-k \leqslant k$. In this case, the sides $A'D'$ and $B'C'$ will be called "short". In the case $k \leqslant n-k$, on the contrary, the sides $A'D'$ and $B'C'$ will be called "long" sides, and the sides $A'B'$ and $A'D'$ will be called "short".

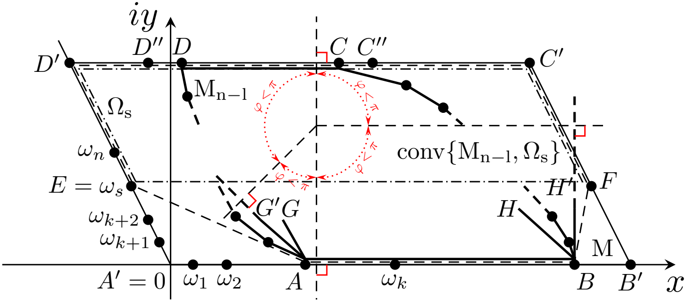

Fig. 4: Sets: M, M_{n-1}, conv$\{M_{n-1}, \text{conv}\,\Omega_s\}$

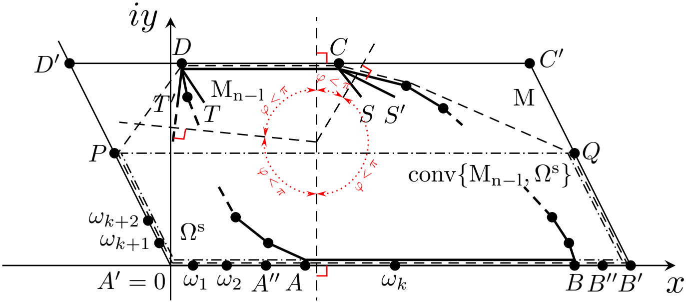

Fig. 5: Sets: M, M_{n-1}, conv$\{M_{n-1}, \text{conv}\,\Omega^s\}$

To formulate the obtained result about the multiple RF completeness for the pencil $L^1(\lambda)$, we will also need the polygon M_{n-l}, which is the convex hull of those points of the set Ω that are sums of different characteristics in the amount of exactly $n-l$ terms.

In the figures 4 and 5 the set M_{n-l} is a polygon $ABCD$, the boundary of which is indicated by a solid thick line. In the case $n-k \leqslant k$, the vertices A and B lie on the «long» side $A'B'$ of the parallelogram M. Vertices C and D lie on the «long» side $D'C'$ of parallelogram M. Lines AD and BC are broken line.

The vertices of the polygon M_{n-l} lying on the "long" sides of the parallelogram will be called the *main vertices*.

It is shown that in the case of splitting boundary conditions (40)–(41) the inclusion $M_\Delta \subset M_{n-l}$ is valid.

It follows from the condition (51) that the polygon M_{n-l}, and hence the characteristic polygon M_Δ, does not touch the "short" sides of the parallelogram M.

Lemma 6

1. If $n-k<l<k$ and ch.p. M_Δ of the pencil $L^1(\lambda)$ touches the "long" sides of the parallelogram M, then $V_s, W_s \in (\alpha)$ for $s=\overline{k+1,n}$.
2. If $k<l<n-k$ and ch.p. M_Δ of the pencil $L^1(\lambda)$ touches the "long" sides of the parallelogram M, then $V_s, W_s \in (\alpha)$ for $s=\overline{1,k}$.

Theorem 5

If $\min\{k,n-k\}<l<\max\{k,n-k\}$, $m=\min\{k,n-k\}$ and the ch.p. M_Δ of the pencil $L^1(\lambda)$ contains the principal vertices of the polygon M_{n-l}, then the system of its RF is m-multiple complete in $L_2[0,1]$ with possible finite defect.

The results of this section are published in detail in the article [2].

6. Completeness of the RF system of a differential operator generated by the simplest 5-th order differential operator and two-term two-point boundary conditions

In the space $L_2[0,1]$ we consider the linear ordinary differential operator L^0 generated by the simplest d.e. of the fifth order

$$\ell^0(y) := y^{(5)}(x), \quad x \in [0,1], \tag{52}$$

and two-point two-term boundary conditions

$$U_\nu^0(y) := \alpha_\nu y^{(\nu-1)}(0) + \beta_\nu y^{(\nu-1)}(1) = 0, \quad \nu = \overline{1,5}, \tag{53}$$

where $\alpha_\nu, \beta_\nu \in \mathbb{C}$ and $|\alpha_\nu| + |\beta_\nu| > 0$, $\nu = \overline{1,5}$.

The interest in this operator is caused by the fact that with an appropriate choice of the coefficients α_ν and β_ν of the boundary conditions (53) this operator can be Birkhoff regular, weakly irregular, and strongly irregular. That is, this operator is a good model operator for which the classical g.f.'s do not satisfy the condition (α), but for which it is possible to construct suitable generalized g.f.'s satisfying this condition.

The main result of this section is the following theorem.

Theorem 6

Suppose that either $\alpha_\nu \neq 0$, $\nu = \overline{1,5}$, or $\beta_\nu \neq 0$, $\nu = \overline{1,5}$. Then either the RF system of the operator L^0 is complete in the space $L_2[0,1]$, or this operator is degenerate, that is, either it has no proper eigenvalues at all, or it has a finite number of eigenvalues, or all $\lambda \in \mathbb{C}$ are its eigenvalues.

This result was announced in 2002. Later in 2003-2009 a more general result was obtained for a differential operator defined by a differential expression of arbitrary order $n = 2m+1$, $m \in \mathbb{N}$. This result is published in detail in the papers [3,4]. The study of the general case is very cumbersome and because of this its essence may not be very clear. For a better understanding of the idea of the method, it seems convenient to consider the simplest case $n = 5$, which preserves the difficulties of the general case.

The proof of the main result of this section on the RF completeness for the operator L^0 is carried out in accordance with the PMC scheme, but not for the operator L^0, but for the closely related pencil $L^0(\rho) := L^0 + \rho^5 E$, where $\lambda = -\rho^5$, generated by the d.e.

$$\ell^0(y,\rho) := y^{(5)}(x) + \rho^5 y, \quad x \in [0,1], \tag{54}$$

and the same two-point two-term boundary conditions as (53).

All results of previous sections 2, 3 and 4 are applicable to the pencil $L^0(\rho)$ of the form (54) and (53), but instead of the spectral parameter λ there will be a spectral parameter ρ. Instead of the usual RF completeness in $L_2[0,1]$ for the operator L^0, we need to study the multiple RF completeness for the pencil $L^0(\rho)$, from which the RF completeness for the operator L^0 will follow.

The assertion of Theorem 6 follows from the 1-fold RF completeness for the pencil $L^0(\rho)$.

Lemma 7

If the RF system of the pencil $L^0(\rho)$ is 1-fold complete in $L_2[0,1]$, then the RF system of the operator L^0 is also complete in $L_2[0,1]$.

In fact, a stronger result was obtained, namely, 5-fold RF completeness for the pencil $L^0(\rho)$.

Theorem 7

Suppose that either $\alpha_\nu \neq 0$, $\nu = \overline{1,5}$, or $\beta_\nu \neq 0$, $\nu = \overline{1,5}$. Then either the RF system of the pencil $L^0(\rho)$ is 5-fold complete in the space $L_2[0,1]$, or this pencil is degenerate.

We need a more detailed classification of the pencil $L^0(\rho)$ (and, consequently, of the operators L^0) than the classification given in section 2.

Let us denote

$$u_{\nu j}(\rho) := U_\nu^0(e^{\rho\omega_j x}) = \alpha_\nu(\rho\omega_j)^{\nu-1} + \beta_\nu(\rho\omega_j)^{\nu-1}e^{\rho\omega_j} = \\ = \rho^{\nu-1}\big(\alpha_\nu\omega_j^{\nu-1} + \beta_\nu\omega_j^{\nu-1}e^{\rho\omega_j}\big) = \rho^{\nu-1}(v_{\nu j} + w_{\nu j}e^{\rho\omega_j}), \quad (55)$$

where $v_{\nu j} = \alpha_\nu\omega_j^{\nu-1}$, $w_{\nu j} = \beta_\nu\omega_j^{\nu-1}$, $\nu, j = \overline{1.5}$.

Let vectors V_j and W_j are determined by the formulas

$$V_j = \begin{pmatrix} v_{1j} \\ v_{2j} \\ v_{3j} \\ v_{4j} \\ v_{5j} \end{pmatrix} := \begin{pmatrix} \alpha_1 \\ \alpha_2\omega_j \\ \alpha_3\omega_j^2 \\ \alpha_4\omega_j^3 \\ \alpha_5\omega_j^4 \end{pmatrix}, \quad W_j = \begin{pmatrix} w_{1j} \\ w_{2j} \\ w_{3j} \\ w_{4j} \\ w_{5j} \end{pmatrix} := \begin{pmatrix} \beta_1 \\ \beta_2\omega_j \\ \beta_3\omega_j^2 \\ \beta_4\omega_j^3 \\ \beta_5\omega_j^4 \end{pmatrix}, \quad (56)$$

and let

$$\Delta_0 = |V_1\, V_2\, V_3\, V_4\, V_5| \; (:= \det(V_1\, V_2\, V_3\, V_4\, V_5)), \; \Delta_1 = |W_1\, V_2\, V_3\, V_4\, V_5|,$$
$$\Delta_2 = |V_1\, W_2\, V_3\, V_4\, V_5|, \; \dots, \; \Delta_5 = |V_1\, V_2\, V_3\, V_4\, W_5|, \; \dots,$$
$$\Delta_{12} = |W_1\, W_{,2}\; V_3\, V_4\, V_5|, \, \Delta_{13} = |W_1\, V_2\, W_3\, V_4\, V_5|, \, \dots,$$
$$\Delta_{12345} = |W_1\, W_2\, W_3\, W_4\, W_5|. \tag{57}$$

Due to the specific nature of the pencil $L(\rho)$ the following lemma is true.

Lemma 8

The following equalities are true

$$\begin{array}{ccccccccc}
\Delta_{12} & = & \Delta_{23} & = & \Delta_{34} & = & \Delta_{45} & = & \Delta_{15}, \\
\Delta_{123} & = & \Delta_{234} & = & \Delta_{345} & = & \Delta_{145} & = & \Delta_{125}, \\
\Delta_{1} & = & \Delta_{2} & = & \Delta_{3} & = & \Delta_{4} & = & \Delta_{5}, \\
\Delta_{1234} & = & \Delta_{2345} & = & \Delta_{1345} & = & \Delta_{1245} & = & \Delta_{1235}, \\
\Delta_{13} & = & \Delta_{24} & = & \Delta_{35} & = & \Delta_{14} & = & \Delta_{25}, \\
\Delta_{124} & = & \Delta_{235} & = & \Delta_{134} & = & \Delta_{245} & = & \Delta_{135}.
\end{array} \tag{58}$$

Let us mark on the plane (see figure 6) all points 0, ω_j, $\omega_j + \omega_k$ $(j \neq k)$, $\omega_j + \omega_k + \omega_l$ $(j \neq k \neq l)$, $\dots$, $\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 (= 0)$ (for brevity, in the figure the number j denotes the point ω_j, the sum $1 + 2$ denotes the point $\omega_1 + \omega_2$, etc.). As in section 2, we denote the set of such points by Ω.

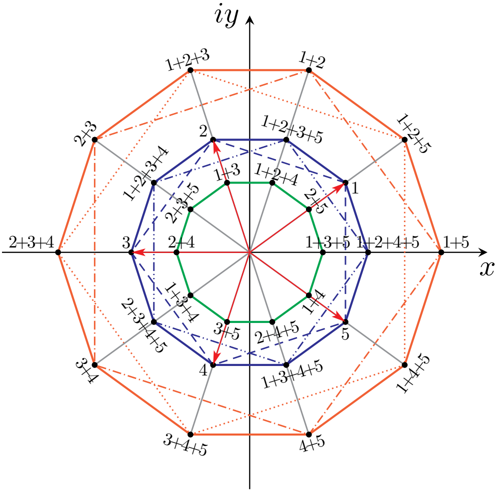

Fig. 6: Polygons M_0, M_0^0, M_0^1, M_1, M_1^0, M_1^1, M_2

Let M_0 be the 10-gon indicated by the red line, M_0^0 and M_0^1 be the two 5-gons tangent to it. Similarly, M_1 is the 10-gon indicated by the blue line, M_1^0 and M_1^1 are the two 5-gons tangent to it. Finally, M_2 is the 10-gon indicated by the green line.

Based on the lemma 8 we obtain the following representation for the characteristic determinant of the pencil $L^0(\rho)$

$$
\begin{gathered}
\Delta(\rho) = \rho^{10}\Big(\Delta_{12}\big(e^{\rho(\omega_1+\omega_2)} + e^{\rho(\omega_2+\omega_3)} + \cdots + e^{\rho(\omega_1+\omega_5)}\big) + \\
+\Delta_{123}\big(e^{\rho(\omega_1+\omega_2+\omega_3)} + e^{\rho(\omega_2+\omega_3+\omega_4)} + \cdots + e^{\rho(\omega_1+\omega_2+\omega_5)}\big) + \\
+\Delta_1\big(e^{\rho\omega_1} + e^{\rho\omega_2} + \cdots + e^{\rho\omega_5}\big) + \\
+\Delta_{1234}\big(e^{\rho(\omega_1+\omega_2+\omega_3+\omega_4)} + e^{\rho(\omega_2+\omega_3+\omega_4+\omega_5)} + \cdots + e^{\rho(\omega_1+\omega_2+\omega_3+\omega_5)}\big) + \qquad (59) \\
+\Delta_{13}\big(e^{\rho(\omega_1+\omega_3)} + e^{\rho(\omega_2+\omega_4)} + \cdots + e^{\rho(\omega_2+\omega_5)}\big) + \\
+\Delta_{124}\big(e^{\rho(\omega_1+\omega_2+\omega_4)} + e^{\rho(\omega_2+\omega_3+\omega_5)} + \cdots + e^{\rho(\omega_1+\omega_3+\omega_5)}\big) + \\
+\Delta_{12345} + \Delta_0\Big).
\end{gathered}
$$

In the figure, we mark the points $\omega_1+\omega_2$, $\omega_2+\omega_3$, $\ldots$, $\omega_5+\omega_1$, if $\Delta_{12} \neq 0$, the points $\omega_1+\omega_2+\omega_3$, $\omega_2+\omega_3+\omega_4$, $\ldots$, $\omega_5+\omega_1+\omega_2$, if $\Delta_{123} \neq 0$, and so on. The convex hull of the marked points M_Δ is the ch.p. of the pencil $L^0(\rho)$ according to the definition given in section 2. Obviously, M_Δ is a polygon, symmetric about the origin and invariant under rotation by an angle of $2\pi/5$. The form of this polygon characterizes the degree of degeneracy of the ch.d.

The following cases are possible (see figure 6):

(**0**) $\Delta_{12} \neq 0 \wedge \Delta_{123} \neq 0$. Here $M_\Delta = M_0$. This is the Birkhoff regular case. The set of pencils $L^0(\rho)$ with this property is denoted by NR_0.

($\mathbf{0^0}$) $\Delta_{12} \neq 0 \wedge \Delta_{123} = 0$. Here $M_\Delta = M_0^0$. This is the first of two weakly irregular cases. The set of pencils $L^0(\rho)$ with this property is denoted by NR_0^0.

($\mathbf{0^1}$) $\Delta_{12} = 0 \wedge \Delta_{123} \neq 0$. Here $M_\Delta = M_0^1$. This is the second of two weakly irregular cases. The set of o.-f. $L^0(\rho)$ with this property is denoted by NR_0^1.

(**1**) $\Delta_1 \neq 0 \wedge \Delta_{12} = \Delta_{123} = 0 \wedge \Delta_{1234} \neq 0$. Here $M_\Delta = M_1$. This is the first of four possible strongly irregular cases. The set of pencils $L^0(\rho)$ with this property is denoted by NR_1.

($\mathbf{1^0}$) $\Delta_1 \neq 0 \wedge \Delta_{12} = \Delta_{123} = \Delta_{1234} = 0$. Here $M_\Delta = M_1^0$. This is the second of four possible strongly irregular cases. The set of pencils $L^0(\rho)$ with this property is denoted by NR_1^0.

($\mathbf{1^1}$) $\Delta_1 = \Delta_{12} = \Delta_{123} = 0 \wedge \Delta_{1234} \neq 0$. Here $M_\Delta = M_1^1$. This is the third of four possible strongly irregular cases. The set of pencils $L^0(\rho)$ with this property is denoted by NR_1^1.

(**2**) $\Delta_1 = \Delta_{12} = \Delta_{123} = \Delta_{1234} = 0$. Here $M_\Delta \subset M_2$. The set of pencils $L^0(\rho)$ with this property is denoted by NR_2. This is the fourth of four possible strongly irregular cases, which contains all the remaining strongly irregular cases. It is shown that all pencils from this set are degenerate.

The central role is played by a very unexpected result about cyclically shifted vectors, which allowed us to solve the problem under consideration.

Let us denote

$$\hat{V}_j = \mathbf{\Omega}^{-1} V_j, \quad \hat{W}_j = \mathbf{\Omega}^{-1} W_j, \quad \mathbf{\Omega} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \omega_1 & \omega_2 & \dots & \omega_5 \\ \dots & \dots & \dots & \dots \\ \omega_1^4 & \omega_2^4 & \dots & \omega_5^4 \end{pmatrix}, \tag{60}$$

$$\hat{\alpha} = (\mathbf{\Omega}^{-1})^T \alpha, \quad \hat{\beta} = (\mathbf{\Omega}^{-1})^T \beta. \tag{61}$$

The following lemma about cyclically shifted vectors is true:

Lemma 9

If $a_k = \hat{\alpha}_k \omega_1^{k-1}$, $b_k = \hat{\beta}_k \omega_1^{k-1}$, $k = \overline{1.5}$ and $\beta_\nu \neq 0$, $\nu = \overline{1,5}$ (for the sake of certainty) then the formulas are valid

$$\hat{V}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}, \ \hat{V}_2 = \begin{pmatrix} a_5 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \ \hat{V}_3 = \begin{pmatrix} a_4 \\ a_5 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ \hat{V}_4 = \begin{pmatrix} a_3 \\ a_4 \\ a_5 \\ a_1 \\ a_2 \end{pmatrix}, \ \hat{V}_5 = \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_1 \end{pmatrix}; \tag{62}$$

$$\hat{W}_1 = (1,0,0,0,0)^T, \quad \hat{W}_2 = (0,1,0,0,0)^T, \quad \dots, \quad \hat{W}_5 = (0,0,0,0,1)^T. \tag{63}$$

Due to the limited time of the report, we will limit ourselves to considering only the strongly irregular case $L^0(\rho) \in \mathrm{NR}_1$.

The following lemma gives an analytical description of this class.

Lemma 10

$L^0(\rho) \in NR_1$ if and only if one of the following two conditions is satisfied:
1) for some value $s \in \mathbb{C}$, $s \neq 0$

$$\theta_{11}(s) = \theta_{21}(s) = \theta_{31}(s) = 0, \quad \theta_{41}(s) \neq 0, \quad \theta_{51}(s) \neq 0, \quad a_5 \neq 0; \tag{64}$$

2) for some value $s \in \mathbb{C}$, $s \neq 0$

$$\theta_{51}(s) = \theta_{11}(s) = \theta_{21}(s) = 0, \quad \theta_{31}(s) \neq 0, \quad \theta_{41}(s) \neq 0, \quad a_4 \neq 0. \tag{65}$$

Lemma 11

If $L^0(\rho) \in \mathrm{N}R_1$ and the condition (64) is satisfied, then the vectors $\mathbf{\Gamma}_j^k = \mathbf{\Omega}\hat{\mathbf{\Gamma}}_j^k$, $j = \overline{1,5}$, and for $k = 1$, and for $k = 2$ are linearly independent and satisfy the condition (α), where
1) in the case $s^5 \neq 1$ (general case):

$$\hat{\Gamma}_1^1 = \begin{pmatrix} 1 \\ s \\ s^2 \\ s^3 \\ s^4 \end{pmatrix}, \ \hat{\Gamma}_2^1 = \begin{pmatrix} s^4 \\ 1 \\ s \\ s^2 \\ s^3 \end{pmatrix}, \ \ldots, \hat{\Gamma}_5^1 = \begin{pmatrix} s \\ s^2 \\ s^3 \\ s^4 \\ 1 \end{pmatrix}. \tag{66}$$

2) in case $s^5 = 1$ (degenerate case), that is, when $s = \varepsilon_j$, $j = \overline{1,5}$, where ε_j are distinct 5th roots of 1,

$$\hat{\mathbf{\Gamma}}_j^2 = (1,\ \varepsilon_j,\ \varepsilon_j^2,\ \varepsilon_j^3,\ \varepsilon_j^4)^T \quad j = \overline{1,5}. \tag{67}$$

Let us consider, for example, the general case 1). Since, based on Lemma 11, there are five linearly independent vector-parameters Γ_j^1 $(j = \overline{1,5})$ satisfying the condition (α), then, based on the previously formulated theorem on the sufficient condition for n-fold completeness (theorem 1, section 4), we obtain 5-fold completeness of the RF system of the pencil $L^0(\rho)$.

Figure 7 shows: the ch.p. M_Δ of the pencil $L^0(\rho)$, denoted by the blue line, and the ch.p. $M(\Gamma_1^1)$ of the parameter vector Γ_1^1, indicated by the purple dashed line. The figure directly shows that $\Gamma_1^1 \in (\alpha)$. The remaining ch.p. $M(\Gamma_j^1)$ $(j = \overline{2,5})$ are obtained from the polygon $M(\Gamma_1^1)$ by successive rotations around the origin by an angle of $2\pi/5$.

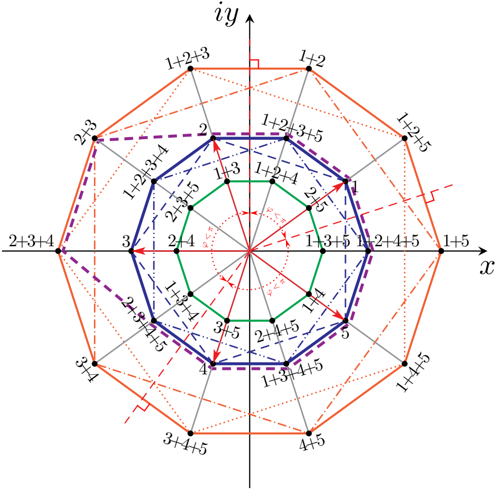

Fig. 7: $M_\Delta = M_1$ $(L^0(\rho) \in \mathrm{NR}_1)$, $M(\Gamma_1^1)$

As already noted at the beginning of the section, in fact, this result is established for the more general ordinary differential operator L^0 generated by the simplest d.e. of the n-th order ($n = 2m+1$, $m \in \mathbb{N}$, $m \geqslant 2$)

$$\ell^0(y) := y^{(n)}(x), \quad x \in [0,1], \tag{68}$$

and two-point two-term boundary conditions

$$U_\nu^0(y) := \alpha_\nu y^{(\nu-1)}(0) + \beta_\nu y^{(\nu-1)}(1) = 0, \quad \nu = \overline{1,n}, \tag{69}$$

where $\alpha_\nu, \beta_\nu \in \mathbb{C}$ and $|\alpha_\nu| + |\beta_\nu| > 0$, $\nu = \overline{1,n}$.

The results of this section are published in detail in the papers [3,4].

7. REFERENCES

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Thank you for your attention!