Spectral Stability in the nonlinear Dirac equation with Soler-type nonlinearity

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Joint work with Aldunate D.[†], Stockmeyer E.[†], Van Den Bosch H.[‡]
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Outline

- Stability for the nonlinear Schrödinger equation
- The Soler model & known results
- Results
- Open questions

Solitary waves

Nonlinear Schrödinger equation: $i\partial_t\psi = -\Delta\psi - f(|\psi|^2)\psi$.
Solitary wave: $\psi(t,x) = e^{-i\omega t}\phi_0(x) \quad \Rightarrow \quad -\Delta\phi_0 - \omega\phi_0 - f(|\phi_0|^2)\phi_0 = 0$.

Existence: characterization in [BL83], with "iif" characterization in 1D.
E.g., for $f(s) = s^\kappa$ in 1D with $\kappa > 0$, existence iif $\omega < 0$.

[BL83] Berestycki & Lions. *Nonlinear scalar field equations. I. Existence of a ground state* (1983). Arch. Ration. Mech. Anal.

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Linearization operator (1D)

Ansatz: $\psi(t,x) = e^{-i\omega t}\left(\phi_0(x) + p(t,x)\right)$

$$\Rightarrow i\partial_t P(t,x) = HP(t,x), \qquad P = \begin{pmatrix} \operatorname{Re} p \\ i \operatorname{Im} p \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix},$$

where $L_- = -\Delta - \omega - f(\phi_0^2)$ and $L_+ = L_- - 2\phi_0^2 f'(\phi_0^2)$.

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Spectral stability

Eigenvalues λ of H verify $\operatorname{Im}\lambda \leqslant 0$.

Allows to characterize orbital stability and (with additional assumptions) implies asymptotic stability.

[CL82] Cazenave & Lions. *Orbital stability of standing waves for nonlinear Schrödinger equations* (1982). Comm. Math. Phys.
[Wei86] Weinstein. *Lyapunov stability of ground states of nonlinear dispersive evolution equations.* (1986). Comm. Pure Appl. Math.
[GSS87] Grillakis, Shatah & Strauss. *Stability of solitary waves in the presence of symmetries, I* (1987). Jour. Funct. An.
[GSS90] Grillakis, Shatah & Strauss. *Stability of solitary waves in the presence of symmetries, II* (1990). Jour. Funct. An.
[Cuc11] Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states* (2011). Comm. Math. Phys.

Spectra for the linearization operators

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Essential Spectrum

$\sigma_{\mathrm{ess}}(L_+) = \sigma_{\mathrm{ess}}(L_-) = \sigma_{\mathrm{ess}}(-\Delta - \omega) = [-\omega, +\infty).$
$\Rightarrow \sigma_{\mathrm{ess}}(H) = (-\infty, \omega] \cup [-\omega, +\infty).$

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Simple eigenvalues

$L_-\phi_0 = 0 \quad$ and $\quad L_+\phi_0' = 0$.
Moreover, $L_- > 0$ on $\{\phi_0\}^\perp$ and L_+ has a single negative eigenvalue.

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In 1D, for $f(s) = s^\kappa$, $\kappa > 0$, V–K criterion equivalent to $\kappa \leqslant 2$.

1D Soler-type model

$$\begin{cases} i\partial_t \psi = D_m \psi - f\left(\langle \psi, \sigma_3 \psi \rangle_{\mathbb{C}^2}\right) \sigma_3 \psi \,, \\ \psi(\cdot, 0) = \phi_0 \in H^1(\mathbb{R}, \mathbb{C}^2) \,. \end{cases}$$

with $f : \mathbb{R} \to \mathbb{R}$, σ_k the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and one-dimensional Dirac operator with mass $m > 0$:

$$D_m = i\sigma_2 \partial_x + m\sigma_3 = \begin{pmatrix} m & \partial_x \\ -\partial_x & -m \end{pmatrix}.$$

[Iva38] Ivanenko, *Notes to the theory of interaction via particles.* (1938). Zh. Éksp. Teor. Fiz
[FLR51] Finkelstein, LeLevier, and Ruderman, *Nonlinear spinor fields.* (1951). Phys. Rev.
[Hei57] Heisenberg, *Quantum theory of fields and elementary particles.* (1957). Physical Review D
[Sol70] Soler, *Classical, stable, nonlinear spinor field with positive rest energy.* (1970). Physical Review D
[GN74] Gross and Neveu, *Dynamical symmetry breaking in asymptotically free field theories.* (1974). Physical Review D

Existence of solitary wave

$$\begin{cases} i\partial_t\psi = D_m\psi - f\left(\langle\psi,\sigma_3\psi\rangle_{\mathbb{C}^2}\right)\sigma_3\psi\,, \\ \psi(\cdot,0) = \phi_0 \in H^1(\mathbb{R},\mathbb{C}^2)\,. \end{cases}$$

[BerCom-12]: Under generic assumption on f (think of $f(s) = s|s|^{p-1}$), solitary wave solutions $\psi(x,t) = e^{-i\omega t}\phi_0(x)$ exists for all $\omega \in (0,m)$.

Initial condition $\phi_0 =: (v,u)^{\mathsf{T}}$ solves

$$L_0\phi_0 := \left(D_m - \omega\,\mathbb{1}\right)\phi_0 - f(\langle\phi_0,\sigma_3\phi_0\rangle_{\mathbb{C}^2})\sigma_3\phi_0 = 0$$

and verifies

- is continuous, decays expon. at rate $\sqrt{m^2-\omega^2}$,
- can be chosen real-valued s.t. v is even with $v(0) > 0$ and u is odd,
- verifies $\langle\psi,\sigma_3\psi\rangle_{\mathbb{C}^2} = v^2 - u^2 > 0$ on $\mathbb{R}$.

[BerCom-12] Berkolaiko and Comech, *On Spectral Stability of Solitary Waves of Nonlinear Dirac Equation in 1D* (2012). Math. Model. Nat. Phenom.
[CV86] Cazenave and Vázquez, *Existence of localized solutions for a classical nonlinear Dirac field.* (1986). Commun. Math. Phys.
[Book-BC19] Boussaïd and Comech, *Nonlinear Dirac equation* (2019). volume 244 of *Mathematical Surveys and Monographs*

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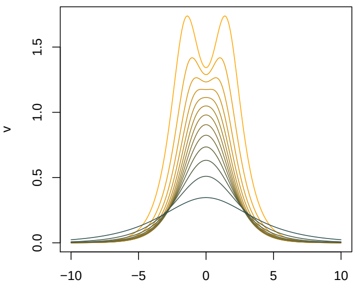

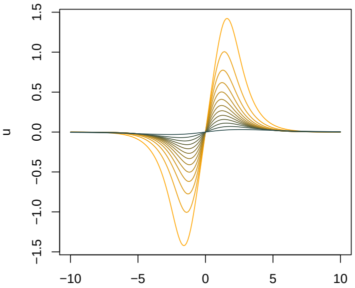

Linearization operator

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where $L_0 \equiv L_0(\omega) := D_m - \omega\,\mathbb{1} - f\left(v^2 - u^2\right)\sigma_3$ and

$$Q := f'(v^2 - u^2)\,\langle \sigma_3 \phi_0,\, \cdot\, \rangle_{\mathbb{C}^2}\, \sigma_3 \phi_0 = f'(v^2 - u^2) \begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix} \geqslant 0.$$

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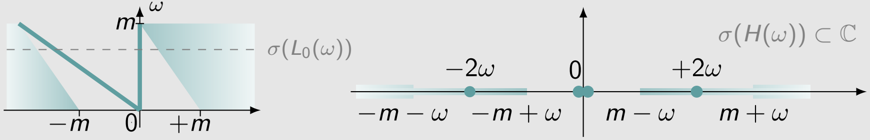
Eigenvalues λ of H verify $\operatorname{Im}\lambda \leqslant 0$.

Spectra for the linearization operators: known properties

$$H = \begin{pmatrix} 0 & L_0 \\ L_2 & 0 \end{pmatrix}, \quad L_0 = D_m - \omega \mathbb{1} - f\left(v^2 - u^2\right)\sigma_3, \quad L_2 = L_0 - 2Q.$$

Known elements of the spectra



Essential spectrum

$\sigma_{\text{ess}}(L_2) = \sigma_{\text{ess}}(L_0) = \sigma_{\text{ess}}(D_m - \omega) = (-\infty, -m-\omega] \cup [m-\omega, +\infty).$
$\Rightarrow \sigma_{\text{ess}}(H) = (-\infty, -m+\omega] \cup [m-\omega, +\infty).$

Simple eigenvalues

$L_0\phi_0 = 0, \quad L_2\phi_0' = 0,$
$L_0\sigma_1\phi_0 = -2\omega\sigma_1\phi_0,$
$Q\sigma_1\phi_0 = 0$

Symmetries

$\sigma(L_0)$ symmetric w.r.t. $-\omega$.
$\sigma(H)$ symmetric w.r.t. the axes $\mathbb{R}$ and $i\mathbb{R}$.
$\Rightarrow$ Spectral stability corresponds to $\sigma(H) \subset \mathbb{R}$.

Spectra for the linearization operators: spectral stability?

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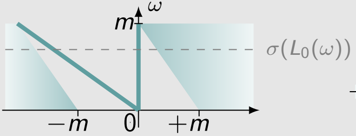

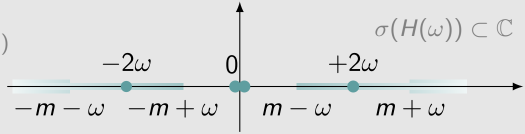

Eigenvalues on axis?

¿z eigenvalue of $H \Rightarrow z^2 \in \mathbb{R}$?

Vakhitov–Kolokolov criterion?

[BCS15] If $\partial_\omega \|\phi_0(\omega)\|_2^2 = 0$, then eigenvalues can *pass through zero* as ω varies.

[BCS15] Berkolaiko, Comech & Sukhtayev. *Vakhitov–Kolokolov and energy vanishing conditions for linear instability of solitary waves in models of classical self-interacting spinor fields.* (2015). Nonlinearity

Spectral stability: known results

Known results for $f(s) = s|s|^{p-1}$

Analytical

- [CGG14]: For $p > 2$, spectral instability when $\omega \to m$.
- [BC16]: For $1 < p \leqslant 2$, spectral stability when $\omega \to m$.

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Numerical

- [BC12, Lak18]: For $p = 1$, spectral stability numerically conjectured (even though debate for small ω's).

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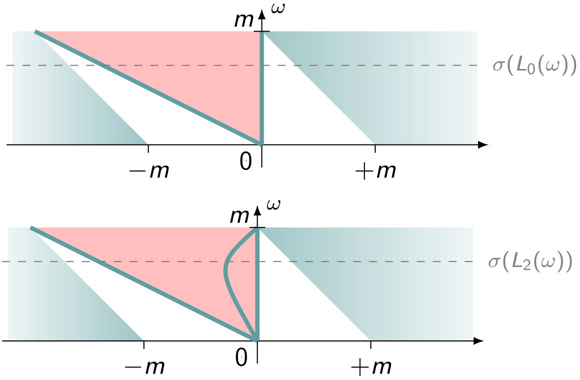
[Lak18] Lakoba, *Numerical study of solitary wave stability in cubic nonlinear Dirac equations in 1d.* (2018). Physics Letters A

Results (I): Groundstates

$$H = \begin{pmatrix} 0 & L_0 \\ L_2 & 0 \end{pmatrix}$$

Theorem (Aldunate–R.–Stockmeyer–Van Den Bosch, 2023)

$L_0 = D_m - \omega \mathbb{1} - f\left(v^2 - u^2\right)\sigma_3$ has no eigenvalues in $(-2\omega, 0)$.
For power nonlinearities, $L_2 = L_0 - 2Q$ has a single eigenvalue in $(-2\omega, 0)$.



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Idea of proof for L_0:

- $(L_0 + \omega)\phi = \lambda\phi$ if and only if

$$\begin{pmatrix} 0 & -\partial_x + M \\ \partial_x + M & 0 \end{pmatrix} \begin{pmatrix} \phi_1 + \phi_2 \\ \phi_1 - \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 + \phi_2 \\ \phi_1 - \phi_2 \end{pmatrix}.$$

  $M := m - f\left(v^2 - u^2\right)$.
- The square is a diagonal matrix with two Schrödinger operators —with essential spectrum $[m^2, +\infty)$— on the diagonal: $-\partial_x^2 + M^2 \mp M'$.
- $v \pm u > 0$ are eigenfunctions of $-\partial_x^2 + M^2 \mp M'$ associated to the same eigenvalue ω^2. By Sturm's oscillation theorem, they are the respective groundstates and $(L_0 + \omega)^2$ has no eigenvalues below ω^2.

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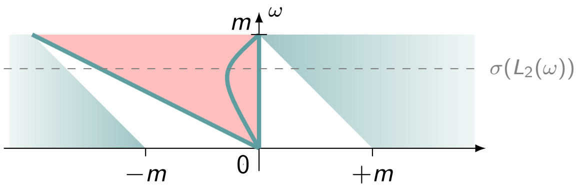
For power nonlinearities, $L_2 = L_0 - 2Q$ has a single eigenvalue in $(-2\omega, 0)$.

Idea of proof for L_2:

- For any f, eigenvalues of $L_2(\omega)$ are simple and continuous in ω.
- For $f(s) = s|s|^{p-1}$, and $\omega \to m$, L_2 has a single eigenvalue in $(-2\omega, 0)$.

  L_2 is self-adjoint with gap in the essential spectrum, its eigenvalues can be characterized variationally [DES00, DES06, SST18].



[DES00] Dolbeault, Esteban, Séré. *On the eigenvalues of operators with gaps.* (2000). J. Funct. Anal.
[DES06] Dolbeault, Esteban, Séré. *General results on the eigenvalues of operators with gaps, arising from both ends of the gaps. Application to Dirac operators.* (2006). J. Eur. Math. Soc.
[SST18] Schimmer, Solovej, Tokus. *Friedrichs Extension and Min–Max Principle for Operators with a Gap.* (2020). Ann. Henri Poincaré.

Results (II): conditions excluding $i\mathbb{R}$-eigenvalues

$$H_\mu := \begin{pmatrix} 0 & L_0 \\ L_0 - \mu Q & 0 \end{pmatrix} = \begin{pmatrix} 0 & L_0 \\ L_\mu & 0 \end{pmatrix}.$$

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Basic properties ($\mu \in \mathbb{R}$)

- $L_\mu \sigma_1 \phi_0 = -2\omega \sigma_1 \phi_0$.
- $\sigma_{\mathrm{ess}}(L_\mu) = (-\infty, -m - \omega] \cup [m - \omega, +\infty)$.
- $\sigma(H_\mu)$ symmetric w.r.t. the real and imaginary axes.
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Assume that $\omega \in (0, m)$ and f are such that

i) L_2 has a single eigenvalue in $(-2\omega, 0)$,

ii) $\partial_\omega \|\phi_0(\omega)\|^2_{L^2} \leqslant 0$ (Vakhitov–Kolokolov criterion).

Then, for any $\mu \in (0, 2)$ the algebraic multiplicity of zero, as an eigenvalue of H_μ, equals 2.

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Corollary (Aldunate–R.–Stockmeyer–Van Den Bosch, 2023)

If in addition, for $\mu \in [0, 2]$, $\operatorname{Re} z^2 \geqslant 0$ for $z \notin i\mathbb{R}$,
then H_2 has no non-zero eigenvalues on the imaginary axis.

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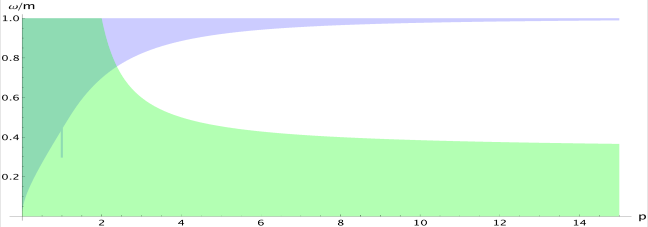
Idea of proof:

- i) $\Leftrightarrow 0 \notin \sigma(L_\mu), \forall \mu \in (0,2) \Rightarrow h(\mu) := \langle \phi_0, L_\mu^{-1}\phi_0 \rangle$ well-defined on $(0,2)$
- h is non-decreasing
- $h(2) = \frac{1}{2}\partial_\omega \|\phi_0(\omega)\|^2_{L^2}$.
- $m_a(0, H_\mu) \geqslant 3 \Leftrightarrow h(\mu) = 0$.

Results (III): no $i\mathbb{R}$-eigenvalues

Theorem (Aldunate–R.–Stockmeyer–Van Den Bosch, 2023)

Let $f(s) = s\,|s|^{p-1}$. H_2 has no non-zero eigenvalues on the imaginary axis for (p, ω) in the intersection of the blue and the green regions.

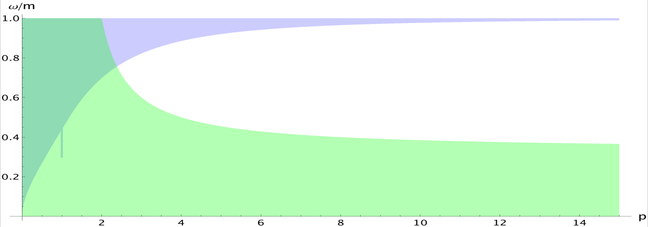


Idea of proof for V–K criterion $\partial_\omega \|\phi_0(\omega)\|^2_{L^2} \leqslant 0$: explicit formula for $\|\phi_0\|^2_{L^2}$.

Results (III): no $i\mathbb{R}$-eigenvalues

Theorem (Aldunate–R.–Stockmeyer–Van Den Bosch, 2023)

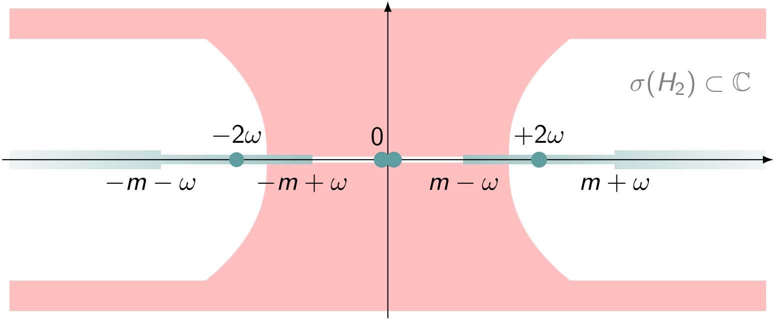
Let $f(s) = s\,|s|^{p-1}$. H_2 has no non-zero eigenvalues on the imaginary axis for (p, ω) in the intersection of the blue and the green regions.



Idea of proof for the bound $\mathrm{Re}(z^2) \geqslant 0$: Assume $H_\mu \psi = z\psi$ with $z \notin \mathbb{R} \cup i\mathbb{R}$

- $\langle \psi_1, L_\mu \psi_1 \rangle = z^2 \left\langle \psi_1, L_0^{-1} \psi_1 \right\rangle$
- $\psi_1 \perp \phi_0$, $\psi_1 \perp \sigma_1 \phi_0$: $\psi_1 = \psi_+ + \psi_-$
- Key identity: $\mathrm{Re}(z^2) = \dfrac{\langle \psi_+, L_\mu \psi_+ \rangle - \langle \psi_-, L_\mu \psi_- \rangle}{\left\langle \psi_+, L_0^{-1} \psi_+ \right\rangle - \left\langle \psi_-, L_0^{-1} \psi_- \right\rangle}$

Results: conclusion



Open problems

- Absence of eigenvalues outside $\mathbb{R} \cup i\mathbb{R}$.
- L_2 has a single eigenvalue in $(-2\omega, 0)$ for general f.
- Orbital / Asymptotic stability.
- Dimension 2, 3: groundstates, spectrum,...

THANK YOU !

D. Aldunate, J. R., E. Stockmeyer, and H. Van Den Bosch, *Results on the Spectral Stability of Standing Wave Solutions of the Soler Model in 1-D*. Commun. Math. Phys. **401**, 227–273 (2023).