Different variants of generalised operator norm convergence

Hello

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Analysis and Mathematical Physics

Motivation I: Approximation of Laplace-like operators

Laplace operators

- on domains (with boundary) or manifolds
- describe waves or heat conduction
- spectrum (e. g. eigenvalues) describe frequency or speed of heat conduction

Two points of view:

- A: Perturbation of a simple problem (limit is simpler)
- or B: Approximations of complicatd problems (approximation is simpler)
- Question: how is the behaviour of the spectrum and related quantities under perturbation or approximations?

Motivation II: classical resolvent convergence

Mathematical formulation:

- Laplace-like operators $\Delta_n (\geq 0)$ in Hilbert space $\mathscr{H}_n = \mathsf{L}_2(X_n)$, (typically) unbounded ($n \in \overline{\mathbb{N}} := \{1, 2, 3, \dots\} \cup \{\infty\}$)
- How to define $\Delta_n \to \Delta_\infty$?

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- How to define $\Delta_n \to \Delta_\infty$?
- Δ_n unbounded? Use resolvents, here $-1 \notin \sigma(\Delta_n)(= \{ z \in \mathbb{C} \mid (\Delta_n - z) \text{ bijektive}\})$, as $\Delta_n \geq 0$, so resolvent $R_n := (\Delta_n + 1)^{-1}$ is bounded operator in $\mathscr{H}$
- $\Delta_n \to \Delta_\infty$ in norm/strong resolvent sense, if $R_n \to R_\infty$ in operator norm/strongly (pointwise):

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Definition (classical norm-/strong resolvent convergence)

- $R_n \xrightarrow{\text{nr}} R_\infty$, if $\|R_n - R_\infty\|_{\mathscr{B}(\mathscr{H})} \leq \delta_n \to 0 \quad (n \to \infty)$ (δ_n convergence speed).
- $R_n \xrightarrow{\text{sr}} R_\infty$, if $\|R_n f - R_\infty f\|_{\mathscr{H}} \to 0$ $(n \to \infty)$ for all $f \in \mathscr{H}$.

Motivation III: Consequences of classical resolvent convergence

- $\Delta_n \geq 0$ unbounded in Hilbert space $\mathscr{H}$
- resolvent $R_n := (\Delta_n + 1)^{-1}$ bounded

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Consequences of norm resolvent convergence:

Theorem (Kato, Reed-Simon, ...: norm resolvent convergence)

If $\Delta_n \xrightarrow{\text{nr}} \Delta_\infty$, then e.g.

- $\|\varphi(\Delta_n) - \varphi(\Delta_\infty)\| \leq C_\varphi \delta_n$ *(z. B. $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$, $\varphi = \mathbb{1}_I$)*
- $\sigma(\Delta_n) \to \sigma(\Delta_\infty)$ *on compact intervals,*
  $$\lambda_\infty \in \sigma(\Delta_\infty) \iff \exists (\lambda_n)_n \colon \lambda_n \in \sigma(\Delta_n), \lambda_n \to \lambda_\infty$$
- *Convergence also for discrete and essential spectrum*

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Consequences of norm resolvent convergence:

Theorem (Kato, Reed-Simon, . . . : strong resolvent convergence)

If $\Delta_n \xrightarrow{\text{sr}} \Delta_\infty$, then e.g.

- $\forall f \in \mathscr{H}$: $\|\varphi(\Delta_n)f - \varphi(\Delta_\infty)f\| \to 0$ *(z. B. $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$, $\varphi = \mathbb{1}_I$)*
- *we only have*

$$\lambda_\infty \in \sigma(\Delta_\infty) \implies \exists(\lambda_n)_n \colon \lambda_n \in \sigma(\Delta_n), \lambda_n \to \lambda_\infty$$

- *Convergence also for discrete spectrum*

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Consequences of norm resolvent convergence:

Theorem (Kato, Reed-Simon, . . . : strong resolvent convergence)

If $\Delta_n \xrightarrow{\text{sr}} \Delta_\infty$, *then e.g.*

- $\forall f \in \mathscr{H}$: $\|\varphi(\Delta_n)f - \varphi(\Delta_\infty)f\| \to 0$ *(z. B.* $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$, $\varphi = \mathbb{1}_I$*)*
- *spectrum can suddenly collaps*

$$\lambda_\infty \in \sigma(\Delta_\infty) \not\Longleftarrow \exists (\lambda_n)_n \colon \lambda_n \in \sigma(\Delta_n), \lambda_n \to \lambda_\infty$$

- *we might have „spectral pollution“*

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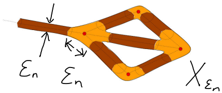
What to do if not only the operator, but also the underlying spaces are changing?

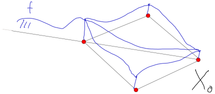
Examples . . .

A: Perturbation of a simple problem

$\mathscr{H}_n = \mathsf{L}_2(X_{\varepsilon_n})$ complicated, $\varepsilon_n = \varepsilon \to 0$, $\mathscr{H}_\infty = \mathsf{L}_2(X_0)$ simpler:

- 

  [P:06, P-Exner:05, 07, 09, 13]

  Δ_n (Neumann) Laplacian on X_{ε_n} $\Delta_\infty f = -f''$ (on each edge)

$\delta_n = O(\varepsilon_n^{1/2})$

$6(\ldots) \sim \varepsilon_n$

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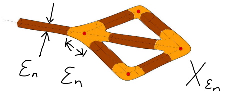
- 

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  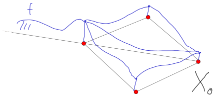

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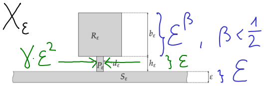
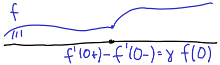
- 

  [Khrabusovskiy-P:21]

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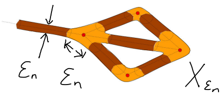
- 

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  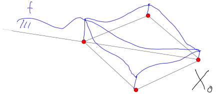

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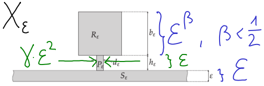
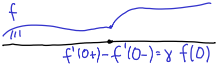
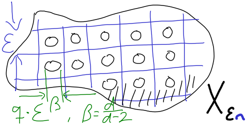
- 

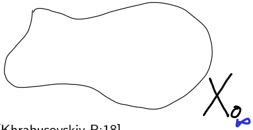
  [Khrabusovskiy-P:21]

- Homogenisation

  Δ_n Dirichlet Laplacian

  $\Delta_\infty = \Delta^{\mathrm{Dir}}_{X_0} + q$ Dirichlet Laplacian on X_0

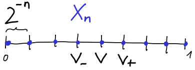
  

  [Khrabusovskiy-P:18]

B: Approximation of complicated problems

Discretisation: $\mathscr{H}_n = \ell_2(X_n, \mu_n)$ simpler, $\mathscr{H}_\infty = \mathsf{L}_2(X_\infty)$ complicated

- simplest example
  - $X_n = \{\, k2^{-n} \mid k = 0, \dots, 2^n \,\} \subset X_\infty = [0, 1]$
  - Discrete Laplacian
    $(\Delta_n f)(v) = 4^n(2f(v) - f(v_+) - f(v_-)), \qquad v \in X_n,\ v_\pm = v \pm 2^{-n}$
  - $\Delta_\infty f = -f''$ (Neumann)



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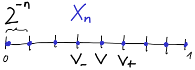
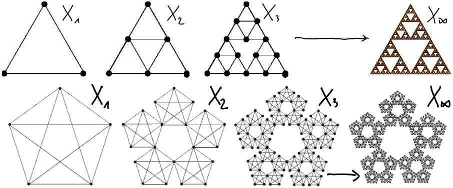


- Fractal



[P-Simmer:18,19,21]

Generalised resolvent convergence: domain perturbations

Let $X_n \subset X = \mathbb{R}^d$ with smooth boundary for $n \in \overline{\mathbb{N}} = \{1, 2, 3, \dots\} \cup \{\infty\}$.

- Let $X_n \to X_\infty$ (e.g. homogenisation)
- Let $\Delta_n \geq 0$ be Dirichlet Laplacian on X_n
- How to define $\Delta_n \to \Delta_\infty$ resp. convergence for resolvents $R_n := (\Delta_n + 1)^{-1}$?

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Idea: use canonical isometry $\iota_n \colon \mathscr{H}_n = \mathsf{L}_2(X_n) \to \mathscr{H} = \mathsf{L}_2(X)$, $\iota_n f_n = f_n \oplus 0$ (extension by 0). We have $\iota_n^* f = f\restriction_{X_n}$.

Definition (Generalised norm resolvent convergence/„Weidmann convergence")

$\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty$, iff $\|\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*\|_{\mathscr{B}(\mathscr{H})} \to 0$ with

$\mathscr{H} = \mathsf{L}_2(X)$ (parent Hilbert space).

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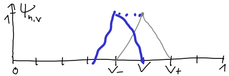
Simple case: If $X_n \subset X_\infty$, use $\mathscr{H} := \mathsf{L}_2(X_\infty)$.

Generalised resolvent convergence: discretisation

- X_n discrete subset of X_∞ (e.g. interval or Sierpiński gasket)

- $\mathcal{E}_n(f) = \sum_{v,v' \in X_n} \underbrace{\gamma_{n,vv'}}_{=2^n \text{ (for interval)}, v \sim v'} |f(v) - f(v')|^2$, $\mathcal{E}_\infty(f) = \int_{X_\infty} |\nabla f|^2 \, \mathrm{d}\mu_\infty$



- $\psi_{n,v} \colon X_\infty \to [0,1]$, Partition of unity $\sum_{v \in X_n} \psi_{n,v} = 1$ on X_∞

- "smoothing":
  $J_n \colon \mathscr{H}_n = \ell_2(X_n, \mu_n) \to \mathscr{H}_\infty = \mathrm{L}_2(X_\infty, \mu_\infty)$,
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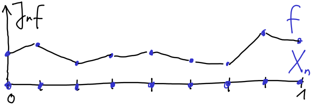

- "discretise"
  $\rightsquigarrow \quad J_n^* \colon \mathrm{L}_2(X_\infty, \mu_\infty) \to \ell_2(X_n, \mu_n)$,
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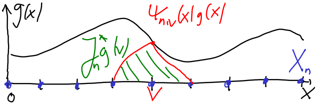

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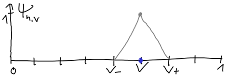

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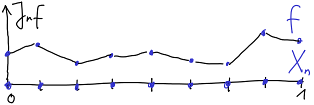

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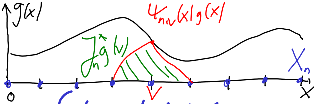

$\mu_n(v) = \int \psi_{n,v}(x) \, dx$

How close is J_n to a unitary operator?

Generalised resolvent convergence: discretisation II

- $J_n \colon \ell_2(X_n, \mu_n) \to \mathsf{L}_2(X_\infty, \mu_\infty)$, $J_n f = \sum_{v \in X_n} f(v)\psi_{n,v}$

$$\rightsquigarrow \; J_n^* \colon \mathsf{L}_2(X_\infty, \mu_\infty) \to \ell_2(X_n, \mu_n), \; (J_n^* g)(v) = \frac{1}{\mu_n(v)} \int_{X_\infty} g \psi_{n,v} \, \mathrm{d}\mu_\infty$$

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How close is J_n to a unitary operator? (see [PS21])

- $(\psi_{n,v})_v$ ist Partition of unity auf X_∞:

$$f(v) = \frac{1}{\mu_n(v)} \sum_{v' \in V} f(v) \langle \psi_{n,v}, \psi_{n,v'} \rangle \quad \left(\text{note } \mu_n(v) := \int_{X_\infty} \psi_{n,v} \, \mathrm{d}\mu_\infty\right)$$

- $(J_n^* J_n f)(v) = \dfrac{1}{\mu_n(v)} \displaystyle\sum_{v' \in X_n} f(v') \langle \psi_{m,v}, \psi_{m,v'} \rangle$ (Partition of unity),
- $(f - J_n^* J_n f)(v) = \dfrac{1}{\mu_n(v)} \displaystyle\sum_{v' \in X_n} (f(v) - f(v')) \langle \psi_v, \psi_{m,v'} \rangle$
- $\rightsquigarrow \|f - J_n^* J_n f\|^2_{\ell_2(G_n, \mu_n)} \le \underbrace{\dfrac{2 \max_v \mu_n(v)}{\min_{vv'} \gamma_{n,vv'}}}_{= 2 \cdot 4^{-n} \to 0 \text{ (for interval)}} \mathcal{E}_n(f).$

Generalised resolvent converence: quasi-unitary equivalence

$\Delta_n \geq 0$ in Hilbert space $\mathscr{H}_n$ for all $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Definition (generalised norm resolvent converence, QUE-convergence, [Pos06, Pos12])

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$:⇔ there is $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ bounded and $\delta_n \to 0$ such that

$$\|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\| \leq \delta_n \qquad \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\| \leq \delta_n, \tag{1}$$

$$\|R_\infty J_n - J_n R_n\| \leq \delta_n \qquad (R_n := (\Delta_n + 1)^{-1}). \tag{2}$$

If (1)–(2) for some J_n, then Δ_n, Δ_∞ are called δ_n-quasi-unitary equivalent.

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$$\|R_\infty J_n - J_n R_n\| \leq \delta_n \qquad (R_n := (\Delta_n + 1)^{-1}). \tag{2}$$

If (1)–(2) for some J_n, then Δ_n, Δ_∞ are called δ_n-quasi-unitary equivalent.

Generalised resolvent converence: quasi-unitary equivalence

$\Delta_n \geq 0$ in Hilbert space $\mathscr{H}_n$ for all $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Definition (generalised norm resolvent converence, QUE-convergence, [Pos06, Pos12])

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$:⇔ there is $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ bounded and $\delta_n \to 0$ such that

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If (1)–(2) for some J_n, then Δ_n, Δ_∞ are called δ_n-quasi-unitary equivalent.

Generalisation of classical norm resolvent convergence

- $\delta_n = 0$ for (1): J_n unitary; w.l.o.g. $\mathscr{H}_n = \mathscr{H}_\infty$, $J_n = \mathrm{id}$.
  Then (2) $\iff$ $\|R_n - R_\infty\| \to 0$ (classical norm resolvent convergence)

Generalised resolvent converence: quasi-unitary equivalence

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If (1)–(2) for some J_n, then Δ_n, Δ_∞ are called δ_n-quasi-unitary equivalent.

Generalisation of unitary equivalence

- $\delta_n = 0$ for (1)–(2): J_n unitary and Δ_n, Δ_∞ unitarily equivalent

Consequences of generalised norm resolvent convergence

Definition (generalised norm resolvent converence, QUE-convergence)

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$:⇔ there is $J \colon \mathscr{H}_\infty \to \mathscr{H}_n$ bounded and $\delta_n \to 0$ such that
(1) $\|(\mathrm{id}_\infty - J^*J)R_\infty\| \le \delta_n$, $\|(\mathrm{id}_n - JJ^*)R_n\| \le \delta_n$, (2) $\|R_n J - JR_\infty\| \le \delta_n$.

Theorem ([Pos06, Pos12])

From $\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$ *we conclude*

- $\|\varphi(\Delta_n) - J_n\varphi(\Delta_\infty)J_n^*\| \le C_\varphi \delta_n$ *(e.g.* $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$, $\varphi = \mathbb{1}_I$*)*
- $\sigma(\Delta_n) \to \sigma(\Delta_\infty)$ *on compact intervals [also for discrete and essential spectrum], convergence of eigenfunctions*

In particular: no spectral pollution

$$\lambda_\infty \in \sigma(\Delta_\infty) \iff \exists (\lambda_n)_n \colon \lambda_n \in \sigma(\Delta_n), \lambda_n \to \lambda_\infty$$

we cannot have (generalised) norm resolvent converence for compact spaces (with purely discrete spectrum) converging towards a non-compact one (with essential spectrum)

Both concepts together

Definition (QUE-convergence)

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$ with speed $\delta_n \to 0$ if $\exists$ contractions $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ s. th.

$$\|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\| \le \delta_n, \ \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\| \le \delta_n, \ \|J_n R_n - R_\infty J_n\| \le \delta_n.$$

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Weidmann [Wei00] (see also [Bög18]) defined for operators acting in different Hilbert spaces: (more precisely for subspaces $\mathscr{H}_n \subset \mathscr{H}$)

Definition (Weidmann convergence)

$\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty$ with speed $\delta_n \to 0$ if there are isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ ($n \in \overline{\mathbb{N}}$) into a Hilbert space with

$$\|\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*\| \le \delta_n.$$

Both concepts together

Definition (QUE-convergence)

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$ with speed $\delta_n \to 0$ if $\exists$ contractions $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ s. th.

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Interestingly, both concepts are equivalent!

Compare both concepts

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$ iff $\|(\mathrm{id}_n - J_n^* J_n)R_n\| \le \delta_n$, $\|(\mathrm{id}_\infty - J_n J_n^*)R_\infty\| \le \delta_n$, $\|J_n R_n - R_\infty J_n\| \le \delta_n$.

$\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty$ iff $\|D_n\| \le \delta_n \to 0$, $D_n := \iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*$, $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ isometries

Theorem ([PZ22])

$$\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty \iff \Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$$

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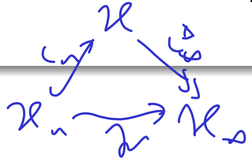
Simple direction: "$\Longrightarrow$".

Step 1: Set $J_n := \iota_\infty^* \iota_n$, $P_n := \iota_n \iota_n^*$, $P_n^\perp = \mathrm{id}_{\mathscr{H}} - P_n$ $(n \in \overline{\mathbb{N}})$ ONP in $\mathscr{H}$

$$(\mathrm{id}_n - J_n^* J_n)R_n = \iota_n^* P_\infty^\perp D_n \iota_n, \qquad (\mathrm{id}_\infty - J_n J_n^*)R_\infty = -\iota_\infty^* P_n^\perp D_n \iota_\infty$$

$$R_n J_n - J_n R_\infty = \iota_\infty^* D_n \iota_n$$



$P_n^\perp \iota_n = 0$

□

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$$R_n J_n - J_n R_\infty = \iota_\infty^* D_n \iota_n$$

$\|D_n\| \le \delta_n$ then Δ_∞ δ_n-QUE (same convergence speed) □

Idea of proof, more complicated direction

$\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$ iff $\|(\mathrm{id}_n - J_n^* J_n) R_n\| \le \delta_n$, $\|(\mathrm{id}_\infty - J_n J_n^*) R_\infty\| \le \delta_n$, $\|J_n R_n - R_\infty J_n\| \le \delta_n$.

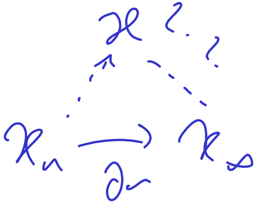
$\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty$ iff $\|D_n\| \le \delta_n \to 0$, $D_n := \iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*$, $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ isometries

Theorem ([PZ22])

$$\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty \iff \Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$$

Complicated direction: "$\Longleftarrow$".

- Find common ("parent") space $\mathscr{H}$ and isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$, that factorise $J_n = \iota_\infty^* \iota_n$ ($n \in \overline{\mathbb{N}}$) (Step 3 later)



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- write $D_n = P_\infty^\perp D_n P_n + P_\infty D_n P_n^\perp + P_\infty D_n P_n$ in terms of

  $(\mathrm{id}_\infty - J_n J_n^*)R_n$, $(\mathrm{id}_n - J_n^* J_n)R_\infty$ and $J_n R_n - R_\infty J_n$ (or only estimate in norm!)
- Step 2: Assume there are isometries ι_n such that $J_n = \iota_\infty^* \iota_n$ (necessarily $\|J_n\| \le 1$) then

$$\|P_\infty^\perp D_n P_n\| = \|R_n(\mathrm{id}_n - J_n^* J_n)R_n\|^{1/2} \le \delta_n^{1/2}, \ldots$$
$$\|P_\infty D_n P_n\| = \|J_n R_n - R_\infty R_n\| \le \delta_n$$

□

How to factorise identification operator?

Define so-called defect operator

$$W_n := (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^{1/2}$$

(idea from Béla Szőkefalvi-Nagy [SNFBK10])

Why?

We have

$$\|J_n f_n\|^2_{\mathscr{H}_\infty} + \|W_n f_n\|^2_{\mathscr{H}_n} = \|f_n\|^2_{\mathscr{H}_n}$$

$$\|P_\infty^\perp D_n P_n\|^2_{\mathscr{B}(\mathscr{H})} = \|W_n R_n\|^2_{\mathscr{B}(\mathscr{H}_n)} = \|R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathscr{B}(\mathscr{H}_n)}$$

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Set ($f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H}$)

$$\mathscr{H} := \mathscr{H}_\infty \oplus \bigoplus_{n \in \mathbb{N}} \mathscr{H}_n$$

$$\iota_\infty f_\infty := (f_\infty, 0, 0, \dots)$$

$$\iota_n f_n := (J_n f_n, 0, \dots, 0, W_n f_n, 0, \dots) \quad (n\text{-th position})$$

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Define so-called defect operator

$$W_n := (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^{1/2}$$

$\|J_n\| \le 1$

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- then ι_∞, ι_n are isometries
- and they factorise J_n, as $\iota_\infty^* \iota_n = J_n$ ☺

A modified version of QUE-convergence

One can see that using a modified version of QUE-convergence leads to the same convergence speed in both directions:

Definition (modified QUE-convergence)

$\Delta_n \xrightarrow[\text{QUE}]{\text{mgnrc}} \Delta_\infty$:⇔ there are contractions $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ and $\delta_n \to 0$ such that

$$\|R_n(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|^{1/2} \le \delta_n, \qquad \|R_\infty(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|^{1/2} \le \delta_n,$$

$$\|R_\infty J_n - J_n R_n\| \le \delta_n \qquad (R_n := (\Delta_n + 1)^{-1}).$$

- Note that $\Delta_n \xrightarrow[\text{QUE}]{\text{gnrc}} \Delta_\infty$ (with δ_n) implies $\Delta_n \xrightarrow[\text{QUE}]{\text{mgnrc}} \Delta_\infty$ (with $\delta_n^{1/2}$)
- the first changed condition of the modified QUE-convergence is equivalent with
  $(0 \le)\|f_n\|^2 - \|J_n f_n\|^2 \le \delta_n^2 \|(\Delta_n + 1) f_n\|^2$ for all $f \in \operatorname{dom} \Delta_n$

Other concepts of convergence for varying spaces

- In homogenisation theory (Marchenko-Khruslov, Pastukhova and many others ...): some sort of strong resolvent convergence
- Γ (or Mosco) convergence: related to strong resolvent convergence ($\|R_n f - R_\infty f\| \to 0$ for all $f \in \mathscr{H}$)

Strong convergence does not imply spectral convergence! (only $\sigma(H) \subset \lim_{\varepsilon \to 0} \sigma(\Delta_n)$ — spectral pollution possible

V. A. Marchenko and E. Y. Khruslov, Homogenization of partial differential equations, Progress in Mathematical Physics, vol. 46, Birkhäuser Boston, Inc., Boston, MA, 2006, Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky.

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Our equivalence of both concepts allows to define strong resolvent convergence also for the QUE-setting. In Weidmann's setting this is clear: $\|\iota_n R_n \iota_n^* f - \iota_\infty R_\infty \iota_\infty^* f\|_{\mathscr{H}} \to 0$ for all $f \in \mathscr{H} \rightsquigarrow$ work in progress . . .

Distances related to the generalised convergences

Describe convergence in terms of a distance (or pseudo-metric)

- consider δ_n as a sort of distance between (self-adjoint) R_1 and R_2 and define

$$d_{\mathrm{iso}}(R_1, R_2) := \inf\{ \|\iota_1 R_1 \iota_1^* - \iota_2 R_2 \iota_2^*\| \mid \iota_n \colon \mathscr{H}_1 \to \mathscr{H} \text{ isom. } \mathscr{H} \text{ Hilbert space} \}$$

$$d_{\mathrm{que}}(R_1, R_2) := \inf\{ \delta(R_1, R_2, J) \mid J \colon \mathscr{H}_1 \to \mathscr{H}_2 \text{ contraction} \}$$

$$\delta(R_1, R_2, J) := \max\{\|R_1(\mathrm{id}_1 - J^* J)R_1\|^{1/2}, \|R_2(\mathrm{id}_2 - JJ^*)R_2\|^{1/2}, \|R_2 J - JR_1\|\}$$

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Clearly we have:

- $\Delta_n \xrightarrow[\text{Weid}]{\text{gnrc}} \Delta_\infty \iff d_{\mathrm{iso}}(R_n, R_\infty) \to 0$ and
- $\Delta_n \xrightarrow[\text{QUE}]{\text{mgnrc}} \Delta_\infty \iff d_{\mathrm{que}}(R_n, R_\infty) \to 0$

Distances related to the generalised convergences II

Our next main result is:

Theorem ([PZ24a])

For self-adjoint and bounded operators R_1 and R_2 we have

- $d_{\mathrm{que}}(R_1, R_2) \leq d_{\mathrm{iso}}(R_1, R_2) \leq \sqrt{3} d_{\mathrm{que}}(R_1, R_2)$

Distances related to the generalised convergences II

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For self-adjoint and bounded operators R_1 *and* R_2 *we have*

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- $d_{\mathrm{iso}}(R_1, R_2) = d_{\mathrm{Hausd}}(\sigma(R_1), \sigma(R_2))$, *if both have purely essential spectrum containing* 0

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- $d_{\mathrm{iso}}(R_1, R_2) = 0 \iff d_{\mathrm{que}}(R_1, R_2) = 0 \iff \sigma_\bullet(R_1) = \sigma_\bullet(R_2)$, $\bullet \in \{\mathrm{ess}, \mathrm{disc}\}$

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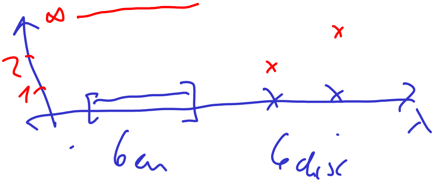
- $d_{\mathrm{iso}}(R_1, R_2) = 0 \iff d_{\mathrm{que}}(R_1, R_2) = 0 \iff \sigma_\bullet(R_1) = \sigma_\bullet(R_2)$, $\bullet \in \{\mathrm{ess}, \mathrm{disc}\}$
- $\Delta_n \xrightarrow[\mathrm{Weid}]{\mathrm{gnrc}} \Delta_\infty \iff d_{\mathrm{iso}}(R_n, R_\infty) \to 0 \iff \Delta_n \xrightarrow[\mathrm{QUE}]{\mathrm{mgnrc}} \Delta_\infty \iff d_{\mathrm{que}}(R_n, R_\infty) \to 0$

Idea of proof

- We use a result by Azoff and Davies [AD84] defining
  $d_{\mathrm{uni}}(R_1, R_2) := \inf\{ \|U_{12} R_1 U_{12}^* - R_2\|_{\mathscr{B}(\mathscr{H}_2)} \mid U_{12} \in \mathrm{Uni}(\mathscr{H}_1, \mathscr{H}_2) \}$,
- Clearly, $d_{\mathrm{uni}}(R_1, R_2) \geq d_{\mathrm{iso}}(R_1, R_2)$

Idea of proof

- We use a result by Azoff and Davies [AD84] defining $d_{\text{uni}}(R_1, R_2) := \inf\{ \, \|U_{12} R_1 U_{12}^* - R_2\|_{\mathscr{B}(\mathscr{H}_2)} \, \big| \, U_{12} \in \text{Uni}(\mathscr{H}_1, \mathscr{H}_2) \, \}$,
- Clearly, $d_{\text{uni}}(R_1, R_2) \geq d_{\text{iso}}(R_1, R_2)$
- Azoff and Davies define the so-called crude multiplicity function $\alpha_{R_n}(\lambda) := \lim_{r \to 0} \operatorname{rank} \mathbb{1}_{(\lambda - r, \lambda + r)}(R_n) \in \bar{\mathbb{N}}_0 := \{0, 1, 2, \dots\} \cup \{\infty\}$.
- We have $\alpha_{R_1} = \alpha_{R_2}$ iff R_1 and R_2 have the same essential and discrete spectrum (including multiplicity)



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- Azoff and Davies show (a sort of Prokhorov distance)

$$\begin{aligned} &\delta(\alpha_{R_1}, \alpha_{R_2}) \\ &:= \inf\{ \, \varepsilon \ge 0 \, | \, \forall I \text{ op. int.: } \ \alpha^*_{R_1}(I) \le \alpha^*_{R_2}(B_\varepsilon(I)) \text{ and } \alpha^*_{R_2}(I) \le \alpha^*_{R_1}(B_\varepsilon(I)) \, \} \\ &\qquad = d_{\mathrm{uni}}(R_1, R_2), \end{aligned}$$

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$\alpha_{R_n} = \alpha_{\iota_n R_n \iota_n^*}$

- Main observation (from us): if $0 \in \sigma_{\mathrm{ess}}(R_1) \cap \sigma_{\mathrm{ess}}(R_2)$ then $\delta(\alpha_{R_1}, \alpha_{R_2}) = \delta(\alpha_{\iota_1 R_1 \iota_1^*}, \alpha_{\iota_2 R_2 \iota_2^*})$ (R_n and $\iota_n R_n \iota_n^*$ differ only in 0 in their spectrum)

Sharpness of our results

- For $R_n = r_n \,\mathrm{id}_{\mathscr{H}_n}$ ($r_n \in \mathbb{R}$) we have

$$d_{\mathrm{que}}(R_1, R_2) = \Big(\frac{1}{|r_1 - r_2|^2} + \frac{1}{\max\{|r_1|^2, |r_2|^2\}}\Big)^{-1/2}$$
$$\le d_{\mathrm{iso}}(R_1, R_2) = \min\{|r_1 - r_2|, \max\{|r_1|, |r_2|\}$$
$$\le d_{\mathrm{uni}}(R_1, R_2) = |r_1 - r_2| = \|R_1 - R_2\|_{\mathscr{B}(\mathscr{H})} = d_{\mathrm{Hausd}}(\sigma(R_1), \sigma(R_2))$$

  - the first inequality is strict if $r_1 \ne r_2$ (note that $(r_1^{-2} + r_2^{-2})^{-1/2} < \min\{r_1, r_2\}$)
  - the second is strict provided $r_1 \cdot r_2 < 0$.
- For $R_1 = R$ and $R_2 = 0$ we have

$$d_{\mathrm{que}}(R, 0) = \frac{1}{\sqrt{2}}\|R\| \le d_{\mathrm{iso}}(R, 0) = d_{\mathrm{uni}}(R, 0) = \|R\|.$$

This shows that the (maybe non-optimal) constant $\sqrt{3}$ has to be at least $\sqrt{2}$.

Outlook

- the quasi-metrics d_{uni}, d_{iso} and d_{que} describe actually distances between the essential spectra and a distance between the discrete spectra (respecting multiplicity)
- some results extend to non-self-adjoint operators
- equivalence of QUE- and Weidmann's convergence allows to transfer strong resolvent convergence to the QUE case
- extension to Banach spaces possible (?)
- refine method with identification operators

$$J_n^1 \colon \operatorname{dom} \mathcal{E}_n \to \operatorname{dom} \mathcal{E}_\infty \quad \text{and} \quad J_n'^1 \colon \operatorname{dom} \mathcal{E}_\infty \to \operatorname{dom} \mathcal{E}_n,$$

on the level of enery form domains such that $\|(J_n - J_n^1)f\| \le \delta_n \|f\|_{\mathcal{E}_n}$ resp. $\|(J_n^* - J_n'^1)u\| \le \delta_n \|u\|_{\mathcal{E}_\infty}$.

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