Discrete and Continuous Non-Self-Adjoint Hill Operators whose Spectrum is a Real Interval

Vassilis G. Papanicolaou

Department of Mathematics
National Technical University of Athens
Zografou Campus, 157 80, Athens, GREECE
papanico@math.ntua.gr

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Abstract

We derive certain properties of the general discrete second-order periodic operator on the integer lattice with complex coefficients. In particular, we investigate the case where the spectrum is an interval of the real line. Recall that in the discrete case there is no Liouville transformation which transforms the general second-order operator to a (discrete) Schrödinger operator.

We also we discuss a conjecture regarding the continuous Hill operator with a complex potential whose spectrum is the positive real axis. Such potentials have some physical significance ($\mathcal{PT}$-symmetry).

1 The Complex Hill Operator

Consider the operator

$$Hy := -y + q(x)y, \qquad x \in \mathbb{R},$$

where $q(x)$ is **complex-valued** and 2π-periodic:

$$q(x + 2\pi) = q(x), \qquad x \in \mathbb{R}.$$

The operator H is acting in $L^2(\mathbb{R})$.

If $q(x)$ is **real-valued** (and locally square-integrable), then H is **self-adjoint**.

There is a huge amount of literature devoted to the self-adjoint case.

The case of a **complex-valued** $q(x)$ is mathematically intriguing and has been studied extensively too (V. Tkachenko, F. Gesztesy, et al.). As expected, the theory is quite different from the self-adjoint case.

The recent emergence of the $\mathcal{PT}$-Symmetric Quantum Theory provides another strong motivation for studying non-self-adjoint Schrödinger operators ("non-Hermitian Hamiltonians" in the physicists' terminology), especially in the case where their spectra are real.

2 Floquet Theory, Discriminant and Spectrum

Consider the problem

$$Hy = -y'' + q(x)y = \lambda y = k^2 y, \qquad x \in \mathbb{R}, \tag{1}$$

where

$$\lambda = k^2 \in \mathbb{C}$$

is the spectral parameter.

Let $u(x) = u(x;\lambda)$ and $v(x) = v(v;\lambda)$ be the solutions of (1) such that

$$u(0;\lambda) = 1, \quad u'(0;\lambda) = 0, \qquad\qquad v(0;\lambda) = 0, \quad v'(0;\lambda) = 1,$$

where primes denote derivatives with respect to x.

The Wronskian of $u(x)$ and $v(x)$ is identically equal to 1.

In particular, $u(x)$ and $v(x)$ are linearly independent functions of x.

Since we have smooth dependence on the parameter λ, the solutions $u(x;\lambda)$ and $v(x;\lambda)$ are entire in λ. Their orders are $\leq 1/2$.

In the case $q(x) \equiv 0$ (the **unperturbed** case) we have

$$\tilde{u}(x;\lambda) = \cos\left(\sqrt{\lambda}\, x\right) \qquad \text{and} \qquad \tilde{v}(x;\lambda) = \frac{\sin\left(\sqrt{\lambda}\, x\right)}{\sqrt{\lambda}}$$

(tilded quantities will be associated to the unpertubed case).

Now let $\mathcal{S}$ be the "shift" or monodromy operator

$$(\mathcal{S}f)(x) := f(x+2\pi).$$

The periodicity of $q(x)$ implies that the linear operator $\mathcal{S}$ maps solutions of (1) to solutions of (1) for the same value of λ (in other words, $\mathcal{S}$ commutes with H), and by exploiting this simple observation one can develop the Floquet/spectral theory of H.

For each $\lambda \in \mathbb{C}$ let $\mathcal{W} = \mathcal{W}(\lambda)$ be the two-dimensional vector space of the solutions of (1). The matrix of the operator $\mathcal{S}|_{\mathcal{W}}$ with respect to the basis (u, v) is

$$S = S(\lambda) = \begin{bmatrix} u(2\pi;\lambda) & v(2\pi;\lambda) \\ u'(2\pi;\lambda) & v'(2\pi;\lambda) \end{bmatrix}$$

(the matrix S and the vector space $\mathcal{W}$ depend on λ).
S is the **Floquet or monodromy matrix** associated to equation (1) and

$$\det S(\lambda) \equiv 1.$$

It follows that the characteristic polynomial of $S(\lambda)$ is

$$\det(S - \rho I) = \rho^2 - \Delta(\lambda)\,\rho + 1,$$

where

$$\Delta(\lambda) := \mathrm{tr} S(\lambda) = u(2\pi;\lambda) + v'(2\pi;\lambda)$$

is the **Hill discriminant** (also known as **Lyapunov's function**) of H.

$\Delta(\lambda)$ is entire of order 1/2.

Sometimes we may find more convenient, instead of λ, to work with the parameter k (recall that $\lambda = k^2$) and, to avoid confusion, whenever we view the discriminant as a function of k, we will denote it by $D(k)$, so that

$$D(k) = \Delta(k^2) = \Delta(\lambda).$$

Clearly, $D(k)$ is an even entire function of order 1.

A remarkable result of V. Tkachenko is that for a function $D(k)$ to be the Hill discriminant of some Hill operator with a 2π-periodic potential $q(x) \in L^2_{\mathrm{loc}}(\mathbb{R})$, it is **necessary and sufficient** that it be an even entire function (of order one) of exponential type 2π which may be represented in the form

$$D(k) = 2\cos(2\pi k) + 2\pi\langle q\rangle \frac{\sin(2\pi k)}{k} - \pi^2\langle q\rangle^2 \frac{\cos(2\pi k)}{k^2} + \frac{h(k)}{k^2}, \qquad k \in \mathbb{C},$$

where

$$\langle q\rangle = \frac{1}{2\pi}\int_0^{2\pi} q(x)\,dx$$

and $h(k)$ is an (even) entire function of order ≤ 1; if the order of $h(k)$ is 1, then its type is $\leq 2\pi$. Furthermore, $h(k)$ satisfies the conditions

$$\int_{-\infty}^{\infty} |h(k)|^2\,dk < \infty \qquad \text{and} \qquad \sum_{n=-\infty}^{\infty} \left| h\left(\frac{n}{2}\right)\right| < \infty.$$

Now, let $\rho_1(\lambda)$ and $\rho_2(\lambda) = \rho_1(\lambda)^{-1}$ be the eigenvalues of $S(\lambda)$, namely the **Floquet multipliers** of H. We have

$$\rho_1(\lambda) + \rho_2(\lambda) = \mathrm{tr}S(\lambda) = \Delta(\lambda),$$

and

$$\rho_1(\lambda), \rho_2(\lambda) = \frac{\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4}}{2}.$$

The eigenvectors of $S(\lambda)$ associated to its eigenvalues $\rho_1(\lambda)$ and $\rho_2(\lambda)$ correspond to the the **Floquet solutions** $\phi_1(x;\lambda)$ and $\phi_2(x;\lambda)$ of (1) satisfying

$$\phi_j(x + 2\pi) = (\mathcal{S}\phi_j)(x) = \rho_j \phi_j(x), \qquad j = 1, 2.$$

Notice that $\rho_1(\lambda) = \rho_2(\lambda)$ can happen only if $\rho_1(\lambda) = \rho_2(\lambda) = \pm 1$ (equivalently, $\Delta(\lambda) = \pm 2$). In this case we may not have two linearly independent Floquet solutions. If two linearly independent Floquet solutions exist, then we say we have **coexistence**.

It is sometimes more convenient to view $\rho_1(\lambda)$ and $\rho_2(\lambda)$ as the two branches of a (single-valued) analytic function $\rho(\lambda)$ defined on the Riemann surface of the function $\sqrt{\Delta(\lambda)^2 - 4}$ (generically this Riemann surface is not compact). Thus,

$$\rho(\lambda) + \frac{1}{\rho(\lambda)} = \Delta(\lambda), \qquad \rho(\lambda) = \frac{\Delta(\lambda) + \sqrt{\Delta(\lambda)^2 - 4}}{2}$$

and $\rho(\lambda)$ can be called the **Floquet multiplier** associated to (1).

The fact that $\Delta(\lambda)$ is entire implies that $\rho(\lambda)$ has neither zeros nor poles (nor essential singularities) for any finite λ. Therefore, the only possible singularities of $\rho(\lambda)$ are square-root branch points at which we must necessarily have $\rho(\lambda) = \pm 1$ (equivalently, $\Delta(\lambda) = \pm 2$).

Actually, $\rho(\lambda)$ must have at least one branch point, since if it had no branch points, then it would have been an entire function of order $\leq 1/2$ with no zeros, therefore, a constant, which is impossible.

In some sense, $\rho(\lambda)$ can be viewed as the analog of the exponential function for the Riemann surface of $\sqrt{\Delta(\lambda)^2 - 4}$. Also,

$$[\log \rho(\lambda)]' = \frac{\rho'(\lambda)}{\rho(\lambda)} = \frac{\Delta'(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}$$

and, since $\rho(\lambda)$ is single-valued on the Riemann surface, we have that the holomorphic differential

$$\frac{\Delta'(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}\, d\lambda$$

has period $2\pi i$ ($\log \rho(\lambda)$ is the **Floquet exponent**).

The values of λ for which $\rho(\lambda) = 1$ (equivalently, $\Delta(\lambda) = 2$) are the **periodic eigenvalues** of H, since, in this case, any associated Floquet solution is 2π-periodic.

The values of λ for which $\rho(\lambda) = -1$ (equivalently, $\Delta(\lambda) = -2$) are the **antiperiodic eigenvalues** of H, since, in this case, any associated Floquet solution is 2π-**antiperiodic**, namely

$$\phi(x + 2\pi) = -\phi(x),$$

As we have already mentioned, $S(\lambda)$ can have a Jordan anomaly only if $\rho(\lambda) = \pm 1$ (equivalently, only if $\Delta(\lambda) = \pm 2$) and in the presence of such an anomaly the matrix $S(\lambda)$ is similar to the Jordan canonical matrix

$$\begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}.$$

Let us mention that $\lambda^\star$ can be a zero of $\Delta(\lambda)^2 - 4$ of even multiplicity, so that $\lambda^\star$ is not a branch point of $\rho(\lambda)$, and, yet, $S(\lambda^\star)$ may not be diagonalizable. If this is the case, we say that the Floquet matrix $S(\lambda)$ has a *pathology of the second kind at* $\lambda^\star$.

If for some $\lambda = \lambda^\star$ we have coexistence of two periodic or, respectively, antiperiodic solutions, then

$$S(\lambda^\star) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{resp.} \quad S(\lambda^\star) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $\lambda^\star$ is a periodic eigenvalue for which we have coexistence of two periodic solutions, then $\lambda^\star$ is a zero of $\Delta(\lambda) - 2$ of multiplicity ≥ 2. Likewise, if $\lambda^\star$ is an antiperiodic eigenvalue for which we have coexistence of two antiperiodic solutions, then $\lambda^\star$ is a zero of $\Delta(\lambda) + 2$ of multiplicity ≥ 2 (algebraic multiplicity $\geq$ geometric multiplicity).

The last statement follows from the formula

$$\begin{aligned} \Delta'(\lambda) =& u(2\pi;\lambda)\int_0^{2\pi} u(x;\lambda)v(x;\lambda)dx - v(2\pi;\lambda)\int_0^{2\pi} u(x;\lambda)^2 dx \\ &+ u'(2\pi;\lambda)\int_0^{2\pi} v(x;\lambda)^2 dx - v'(2\pi;\lambda)\int_0^{2\pi} u(x;\lambda)v(x;\lambda)dx. \end{aligned}$$

2.1 The Spectrum

The spectrum $\sigma(H)$ of H is characterized as

$$\sigma(H) = \{\lambda \in \mathbb{C} : |\rho(\lambda)| = 1\} = \{\lambda \in \mathbb{C} : \rho(\lambda) = e^{i\theta}, \quad 0 \le \theta \le \pi\},$$

$$= \{\lambda \in \mathbb{C} : \Delta(\lambda) \in [-2, 2]\} = \{\lambda \in \mathbb{C} : \Delta(\lambda) = 2\cos\theta, \quad 0 \le \theta \le \pi\}.$$

Notice that $\sigma(H)$ is an unbounded closed subset of $\mathbb{C}$ (this follows, e.g., from the fact that $\Delta(\lambda)$ is entire of order $1/2$ and, consequently, takes every value in $[-2, 2]$ infinitely many times).

More precisely (V. Tkachenko, F. Gesztesy) $\sigma(H)$ is a countable system (i.e. union) of analytic arcs, where the analyticity of such an arc may fail only at a point λ such that $\Delta'(\lambda) = 0$ (while $\Delta(\lambda) = 2\cos\theta$ for some $\theta \in [0, \pi]$, so that λ lies in the spectrum). Furthermore, the resolvent set $\mathbb{C} \setminus \sigma(H)$ of H is path-connected. In particular, $\sigma(H)$ cannot contain closed curves and, also, it cannot be a piecewise analytic curve without an endpoint. Asymptotically, the spectral arcs approach the half-line (the asymptotic form of the spectrum)

$$\ell_{\langle q\rangle} = \{z \in \mathbb{C} \,:\, z = \langle q\rangle + x,\ x \ge 0\}.$$

Observe that if $\lambda^\star$ is a periodic or antiperiodic eigenvalue, then $\Delta(\lambda^\star) = \pm 2$, hence $\lambda^\star \in \sigma(H)$.

3 The Case where $\sigma(H)$ is a Single Analytic Arc

Suppose that the spectrum $\sigma(H)$ is an analytic (connected) curve. Since $\mathbb{C} \setminus \sigma(H)$ is path-connected, $\sigma(H)$ must have an endpoint, say λ_0.

By replacing $q(x)$ by $q(x) - \lambda_0$, we can assume that the endpoint of $\sigma(H)$ is 0.

Suppose $\Delta(\lambda^*)^2 - 4 = 0$. Then $\lambda^* \in \sigma(H)$. Let us assume that $\lambda^* \neq 0$ so that λ^* is an "interior" point of $\sigma(H)$. The Taylor expansion of $\Delta(\lambda)$ about λ^* gives

$$\Delta(\lambda) = \pm 2 + c(\lambda - \lambda^*)^d + O\left[(\lambda - \lambda^*)^{d+1}\right], \qquad \lambda \to \lambda^*,$$

where d is an integer ≥ 1 and $c \neq 0$. Then, the assumption that λ^* is an interior point of $\sigma(H)$ forces $d = 2$, hence λ^* cannot be a branch point of $\rho(\lambda)$.

It follows that 0 is the unique branch point of $\rho(\lambda)$. Thus,

$$\rho(\lambda) = f\left(\sqrt{\lambda}\right) = f(k) \qquad (\text{since } \ \lambda = k^2),$$

where $f(k)$ is entire of order 1 and has no zeros. Furthermore 0 is a branch point of $\rho(\lambda)$ and, hence, $\rho(0) = \pm 1$. Therefore, $\rho(\lambda)$ must be of the form

$$\rho(\lambda) = \pm e^{i\alpha\sqrt{\lambda}},$$

where $\alpha \neq 0$ is a complex constant.

Hence,

$$\Delta(\lambda) = \rho(\lambda) + \rho(\lambda)^{-1} = \pm 2\cos\left(\alpha\sqrt{\lambda}\right),$$

and the general characterization of the discriminant implies that $\alpha = 2\pi$ and $\langle q - \lambda_0 \rangle = 0$ (i.e. for our original $q(x)$ we must have $\langle q \rangle = \lambda_0$). Furthermore,

$$\Delta(\lambda) = 2\cos\left(2\pi\sqrt{\lambda}\right), \qquad \text{hence} \quad \rho(\lambda) = e^{2\pi i\sqrt{\lambda}}$$

and, consequently,

$$\sigma(H) = [0, \infty)$$

(for our original $q(x)$ we must have $\sigma(H) = \langle q \rangle + [0, \infty)$). Notice also that $\rho(0) = 1$, hence 0 is a periodic eigenvalue. Furthermore, $\Delta'(\lambda) = -2\pi \sin\left(2\pi\sqrt{\lambda}\right)/\sqrt{\lambda}$, hence $\Delta'(0) = -4\pi^2 \neq 0$, which implies that for $\lambda = 0$ we cannot have coexistence.

Thus, $S(\lambda)$ does not have a *pathology of the first kind* at $\lambda = 0$ (a pathology of the first kind at $\lambda^\star$ occurs if $\lambda^\star$ is a branch point of $\rho(\lambda)$ and at the same time we have coexistence of two periodic or antiperiodic solutions at $\lambda = \lambda^\star$).

4 The Self-Adjoint Case

In the **self-adjoint case** (i.e. when $q(x)$ is real-valued) $\lambda^\star$ is a double zero of $\Delta(\lambda) - 2$ if and only if we have coexistence of periodic solutions for $\lambda = \lambda^\star$, while $\lambda^\star$ is a double zero of $\Delta(\lambda) + 2$ if and only if we have coexistence of antiperiodic solutions for $\lambda = \lambda^\star$. Furthermore, $\Delta(\lambda)^2 - 4$ cannot have any zeros with multiplicity > 2. In this sense, algebraic multiplicity equals geometric multiplicity. Also, a point $\lambda^\star$ is a branch point of the Floquet multiplier $\rho(\lambda)$ if and only if $S(\lambda^\star)$ has a Jordan anomaly.

The spectrum is a union of closed intervals (the **bands**) separated by open intervals (the **gaps**):

$$\sigma(H) = \bigcup_{n\geq 0} [\lambda_{2n}, \lambda_{2n+1}], \qquad \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \lambda_5 \leq \lambda_6 < \cdots$$

λ_0 and $\lambda_{4j-1} \leq \lambda_{4j}$, $j \leq 1$ are the periodic eigenvalues, while

$\lambda_{4j-3} \leq \lambda_{4j-2}$, $j \leq 1$ are the antiperiodic eigenvalues.

If for some $n \geq 1$ we have that $\lambda_{2n-1} = \lambda_{2n}$,
then the corresponding gap $(\lambda_{2n-1}, \lambda_{2n})$ of the spectrum is **closed** (i.e. empty) and we have coexistence of two linearly independent periodic or antiperiodic solutions.

If $\lambda_{2n-1} < \lambda_{2n}$, then there is no coexistence neither at λ_{2n-1} nor at λ_{2n}.

Clearly, the Dirichlet spectrum $\{\mu_1, \mu_2, \ldots\}$ of H on the interval $(0, 2\pi)$ coincides with the set of (distinct) zeros of the entire function $v(2\pi; \lambda)$.

In the self-adjoint case all the zeros of $v(2\pi; \lambda)$ are simple and, of course, real. Furthermore, if $v(2\pi; \mu) = 0$, then the Floquet matrix at $\lambda = \mu$ becomes

$$S(\mu) = \begin{bmatrix} u(2\pi; \mu) & 0 \\ u'(2\pi; \mu) & v'(2\pi; \mu) \end{bmatrix},$$

hence the **real** quantities $u(2\pi; \mu)$ and $u'(2\pi; \mu)$ are the eigenvalues of $S(\mu)$, i.e. the Floquet multipliers. In particular, $u(2\pi; \mu)u'(2\pi; \mu) = 1$ and, consequently $|\Delta(\mu)| = |u(2\pi; \mu) + u'(2\pi; \mu)| = |u(2\pi; \mu)| + |u'(2\pi; \mu)| \geq 2$.

Actually, we have

$$\lambda_0 < \lambda_1 \leq \mu_1 \leq \lambda_2 < \lambda_3 \leq \mu_2 \leq \lambda_4 < \lambda_5 \leq \mu_3 \leq \lambda_6 < \cdots$$

There is a very short proof of all the above properties of the self-adjoint case. First we check them for the trivial case $q(x) \equiv 0$ and then we consider the continuous deformation of potentials

$$tq(x), \qquad 0 \leq t \leq 1,$$

and exploit the continuous dependence on t (notice that, due to self-adjointness all motion of the λ's and μ's is confined on the real axis).

5 A Well-Known Theorem of Borg

In the famous paper

Borg, G., Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, *Acta Math.*, 78, 1–96 (1946)

among many other inverse spectral results regarding the Sturm-Liouville operator, Borg has shown that for a **real-valued** potential $q(x) \in L^2_{loc}(\mathbb{R})$:

If $\sigma(H) = [0, \infty)$, then $q(x) = 0$ a.e.

Actually, Borg proved a more general statement. He showed that if all the gaps corresponding to antiperiodic eigenvalues are closed, then

$$q(x + \pi) = q(x).$$

QUESTION: Are there analogs or extensions to Borg's theorem in the complex potential case?

It is worth mentioning that Borg's theorem fails in the case where the potential $q(x)$ is quasi-periodic (or limit-periodic?).

6 Gasymov's Discovery

The case of a nonreal $q(x)$, however, is quite different. Gasymov made the remarkable discovery that if

$$q(x) = \sum_{n=1}^{\infty} B_n e^{inx}, \qquad \text{with} \ \sum_{n=1}^{\infty} |B_n| < \infty,$$

then the equation

$$Hy = -y'' + q(x)y = k^2 y,$$

has a Floquet solution of the form

$$\phi(x;k) = e^{ikx} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2k} \sum_{\ell=n}^{\infty} c_{n\ell} e^{i\ell x} \right)$$

where **the coefficients $c_{n\ell}$ do not depend on k** and satisfy

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\ell=n+1}^{\infty} \ell(\ell-n)|c_{n\ell}| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} n|c_{n\ell}| < \infty.$$

It follows that the Floquet multiplier is

$$e^{2\pi ik} = e^{2\pi i\sqrt{\lambda}},$$

and consequently, $\sigma(H) = [0, \infty)$.

Notice also that $\phi(x;k)$ is meromorphic in k whose poles are simple. Furthermore, every pole is of the form $-n/2$, where n is a positive integer and if $k \neq -n/2$, $n = 0, 1, \ldots$, then $\phi(x;-k)$ is the other Floquet solution.

Actually, since the spectral properties of the operator H depend continuously on $q(x)$ with respect to the $L^2(0,2\pi)$-norm, it follows that for the weaker assumption that $\sum_{n=1}^{\infty}|B_n|^2<\infty$ we still have $\sigma(H)=[0,\infty)$.

It is also worth mentioning that there are multidimensional analogs of Gasymov's result (see, e.g., P. Kuchment).

Also,

$$\Delta(\lambda)=2\cos(2\pi\sqrt{\lambda}) \qquad \Rightarrow \qquad \Delta(\lambda)^2-4=-4\sin^2(2\pi\sqrt{\lambda}),$$

hence, the zeros of $\Delta(\lambda)^2-4$ are (counting multiplicities)

$$\left(\frac{n}{2}\right)^2, \qquad n\in\mathbb{Z}.$$

Notice that 0 is a simple zero of $\Delta(\lambda)^2-4$, while all other zeros, namely the zeros $n^2/4$, $n\geq 1$, are double.

Clearly, the only branch point of the Floquet multiplier $\rho(\lambda)=e^{2\pi i\sqrt{\lambda}}$ is $\lambda=0$. However, $S(n^2/4)$ may not be diagonalizable for nonzero values of n (pathology of the second kind).

There is an easy way to (partly) understand Gasymov's result. In the equation

$$-y'' + q(x)y = k^2 y, \qquad\qquad q(x) = \sum_{n=1}^{\infty} B_n e^{inx},$$

we substitute

$$z = e^{ix}, \qquad w(z) = w(e^{ix}) = y(x)$$

Then the equation becomes

$$z^2 w''(z) + z w'(z) + P(z) w(z) = k^2 w(z), \qquad \text{with} \quad P(z) = \sum_{n=1}^{\infty} B_n z^n.$$

This equation has a regular singular point at $z = 0$, therefore its solutions can be expressed in Frobenius series. The indicial equation is

$$r^2 = k^2, \qquad \text{thus} \quad r = \pm k$$

and, hence, the Frobenius solutions are (at least for $k \neq n/2$, $n = 0, \pm 1, \pm 2, \ldots$)

$$w(z) = z^{\pm k} \sum_{n=0}^{\infty} a_n z^n,$$

which implies that the Floquet multiplier of the original equation is $e^{2\pi i k}$ and, consequently, the spectrum is $[0, \infty)$.

7 An Example

For a fixed integer $m \geq 1$ and a fixed complex number $a \neq 0$ with $|a| \neq 1$ we set

$$q_m(x) = \frac{2m^2 a e^{imx}}{(ae^{imx}+1)^2} = \frac{2m^2 a^{-1} e^{-imx}}{(a^{-1}e^{-imx}+1)^2} = \frac{m^2}{2}\operatorname{sech}^2\left(\frac{\xi + imx}{2}\right), \qquad \xi = \ln a.$$

Notice that for $|a| < 1$ we have

$$q_m(x) = 2m^2 a \sum_{n=1}^{\infty} (-1)^{n+1} n e^{inmx}$$

while for $|a| > 1$ we have

$$q_m(x) = 2m^2 a^{-1} \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-inmx}$$

Then, one Floquet solution of the equation

$$-y'' + q_m(x) y = k^2 y,$$

is

$$\phi(x;k) = e^{ikx}\left[1 - \frac{1}{k+(m/2)} \cdot \frac{mae^{imx}}{ae^{imx}+1}\right]$$

(in the case $|a| < 1$ this is the Gasymov solution).

$$\phi(x;k) = e^{ikx}\left[1 - \frac{1}{k+(m/2)} \cdot \frac{mae^{imx}}{ae^{imx}+1}\right]$$

Now, unless $k = m/2$, we have that $\phi(x;-k)$ is also a Floquet solution and, furthermore, $\phi(x;k)$ and $\phi(x;-k)$ are linearly independent for $k \neq 0$ (and $k \neq \pm m/2$). Thus, we have coexistence for all $k \neq 0, \pm m/2$.

For $k = 0$, i.e. for $\lambda = 0$, another solution is

$$\left(x - \frac{4}{im} \cdot \frac{1}{ae^{imx}-1}\right)\phi(x;0),$$

which is, obviously, not periodic. Hence, we do not have coexistence. Furthermore, let us notice that $\lambda = 0$ is a simple zero of $\Delta(\lambda)^2 - 4 = -4\sin^2\left(2\pi\sqrt{\lambda}\right)$.

For $k = \pm m/2$, i.e. for $\lambda = m^2/4$, another solution is

$$\left(2imax + a^2 e^{imx} - e^{-imx}\right)\phi(x;m/2),$$

which is, obviously, neither periodic nor antiperiodic. Hence, again, we do not have coexistence. However, $\lambda = m^2/4$ is a **double** zero of $\Delta(\lambda)^2 - 4 = -4\sin^2\left(2\pi\sqrt{\lambda}\right)$ (pathology of the second kind).

8 A Conjecture

Conjecture. Let $q(x)$ be an entire and 2π-periodic function of x. If the spectrum of the operator $H = -d^2/dx^2 + q(x)$ is

$$\sigma(H) = [0, \infty),$$

then

$$q(x) = \sum_{n=1}^{\infty} A_n e^{-inx} \qquad \text{or} \qquad q(x) = \sum_{n=1}^{\infty} B_n e^{inx}.$$

Terminology. We call *Gasymov potential* any (not necessarily entire) periodic function $G(x)$ whose Fourier series expansion contains only positive or only negative frequencies.

A small indication in favor of the conjecture is the following:

If the Fourier expansion of $q(x)$ contains both positive and negative frequencies, then the resulting equation with respect to $z = e^{ix}$ has a singular singular point at $z = 0$.

9 The Shifted Operator

Let ξ be a given real number. We introduce the shifted operator

$$(H_\xi\, y)(x) = -y''(x) + q_\xi(x)\, y(x) \qquad \text{acting in } \ L^2(\mathbb{R}),$$

where

$$q_\xi(x) = q(x+\xi)$$

(thus $H_0 = H$).

Notation. If A is a quantity associated to the operator H, the corresponding quantity associated to the operator H_ξ will be denoted by A_ξ.

Suppose that $\phi(x;\lambda)$ is a Floquet solution of $Hy = \lambda y$ associated to the Floquet multiplier $\rho(\lambda)$, so that

$$\phi(x+2\pi;\lambda) = \rho(\lambda)\phi(x;\lambda).$$

Then, $\phi(x+\xi;\lambda)$ satisfies the equation $H_\xi\, y = \lambda y$ and we also have that $\phi(x+2\pi+\xi;\lambda) = \rho(\lambda)\phi(x+\xi;\lambda)$, which means that $\phi(x+\xi;\lambda)$ is a Floquet solution of $H_\xi = \lambda y$ associated to the Floquet multiplier $\rho(\lambda)$. And since this is true for every $\lambda \in \mathbb{C}$ it follows that

$$\rho_\xi(\lambda) \equiv \rho(\lambda)$$

i.e. the operators H and H_ξ have the same Floquet multiplier and, consequently,

$$\sigma(H_\xi) = \sigma(H),$$

thus the spectrum of H remains invariant under the shift by ξ.
We also get that

$$\Delta_\xi(\lambda) \equiv \Delta(\lambda), \qquad \text{i.e. } \ u_\xi(2\pi;\lambda) + v_\xi'(2\pi;\lambda) \equiv u(2\pi;\lambda) + v'(2\pi;\lambda).$$

Suppose now that $q(x)$ is analytic in a strip $\mathcal{T}$ of the form $a < \Im(x) < b$ containing the real axis. Then $q_\xi(x) = q(x+\xi)$ makes sense for $\xi \in \mathcal{T}$ and $x \in \mathbb{R}$. Therefore, by analytic continuation the above equations remain true for all $\xi \in \mathcal{T}$. If, in particular, $q(x)$ is entire in x, then they remain true for all $\xi \in \mathbb{C}$.

If, however, $q(x)$ is **meromorphic** in x, the above do not quite hold. For instance, let

$$q(x) = \frac{e^{ix}}{1 - (1/2)\, e^{ix}}.$$

Clearly, $q(x)$ is meromorphic and

$$q(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{2^{n-1}}, \qquad x \in \mathbb{R}.$$

Thus, $q(x)$ is a Gasymov potential and, consequently, $\sigma(H) = [0, \infty)$. Now, let us consider the shifted potential

$$q_\xi(x) = \frac{e^{i\xi} e^{ix}}{1 - (1/2)\, e^{i\xi} e^{ix}}.$$

By choosing $\xi = -i \ln 4$ we get

$$q_\xi(x) = \frac{4e^{ix}}{1 - 2e^{ix}} = \frac{-2}{1 - (1/2)\, e^{-ix}} = -2 - \sum_{n=1}^{\infty} \frac{e^{-inx}}{2^{n-1}}, \qquad x \in \mathbb{R},$$

from which we see that $q_\xi(x) + 2$ is a Gasymov potential and $\sigma(H_\xi) = [-2, \infty) \neq \sigma(H)$.

10 Asymptotic Formulas

Suppose $q(x)$ is in C^2. Then (see, e.g., J. Pöschel and E. Trubowitz),

$$v(x;\lambda) = \tilde{v}(x;\lambda) - \frac{\cos\left(\sqrt{\lambda}\,x\right)}{2\sqrt{\lambda}} Q(x) + \frac{\tilde{v}(x;\lambda)}{4\lambda}\left[q(x) + q(0) - \frac{Q(x)^2}{2}\right] + O\left(\frac{e^{|\Im(\sqrt{\lambda})|x}}{|\lambda|^2}\right), \qquad \lambda \to \infty,$$

where

$$\tilde{v}(x;\lambda) = \frac{\sin\left(\sqrt{\lambda}\,x\right)}{\sqrt{\lambda}} \qquad \text{and} \qquad Q(x) = \int_0^x q(\xi)d\xi.$$

Thus, if

$$\langle q \rangle = \frac{Q(2\pi)}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} q(\xi)d\xi = 0,$$

then

$$v(2\pi;\lambda) = \tilde{v}(2\pi;\lambda) + \frac{\tilde{v}(2\pi;\lambda)}{2\lambda} q(0) + O\left(\frac{e^{2\pi|\Im(\sqrt{\lambda})|}}{|\lambda|^2}\right), \qquad \lambda \to \infty.$$

If N is a sufficiently large integer, then $v(2\pi;\lambda)$ has exactly N zeros (counting multiplicities) in the open half-plane

$$\Re(\lambda) < \left(\frac{N}{2} + \frac{1}{4}\right)^2$$

(notice that $\tilde{v}(2\pi;\lambda)$, too, has exactly N zeros in the above half-plane). Furthermore, for each $n > N$, $v(2\pi;\lambda)$ has exactly one simple zero in the egg-shaped region

$$\left|\sqrt{\lambda} - \frac{n}{2}\right| < \frac{1}{4}$$

and $v(2\pi;\lambda)$ has no other zeros.

11 A Trace Formula

Let $\mu_1, \mu_2, \ldots$ be the zeros of $v(2\pi;\lambda)$ (counting multiplicities) labeled so that $|\mu_1| \le |\mu_2| \le \cdots$. Then, assuming that $q \in C^2$ with

$$\langle q \rangle = \frac{1}{2\pi}\int_0^{2\pi} q(\xi)d\xi = 0,$$

we have the trace formula

$$\lim_n \sum_{j\le n}\left(\mu_j - \frac{j^2}{4}\right) = \sum_{n=1}^{\infty}\left(\mu_n - \frac{n^2}{4}\right) = -\frac{q(0)}{2}.$$

The proof is done by estimating the contour integrals

$$\frac{1}{2\pi i}\oint_{C_n} \lambda \left[\frac{\partial_\lambda v(2\pi;\lambda)}{v(2\pi;\lambda)} - \frac{\partial_\lambda \tilde{v}(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)}\right] d\lambda,$$

where C_n, $n \ge 1$ is the circle of radius $\left(\frac{n}{2}+\frac{1}{4}\right)^2$, centered at 0, while ∂_λ denotes the derivative with respect to λ.

Notice that, for all n sufficiently large the above integral is equal to the sum

$$\sum_{j\le n}\left(\mu_j - \frac{j^2}{4}\right).$$

To estimate the above contour integrals, we start with the asymptotic formula

$$m(\lambda) := \frac{v(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)} = 1 + \frac{q(0)}{2\lambda} + O\left(\frac{1}{\lambda^{5/2}}\right), \qquad \lambda \to \infty, \quad \lambda \in \bigcup_{n=1}^{\infty} T_n,$$

where T_n, $n = 1, 2, \ldots$, are the annuli

$$T_n = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \left(\frac{n}{2} + \frac{1}{4}\right)^2 \right| < 1 + n^{\alpha} \right\}$$

for some fixed $\alpha \in (0, 1)$. Notice that the asymptotic formula also implies

$$\frac{\tilde{v}(2\pi;\lambda)}{v(2\pi;\lambda)} = 1 - \frac{q(0)}{2\lambda} + O\left(\frac{1}{\lambda^{5/2}}\right), \qquad \lambda \to \infty, \quad \lambda \in \bigcup_{n=1}^{\infty} T_n,$$

Next, let $\Gamma \subset T_n$ be the circle of radius n^{α}, centered at $\lambda \in C_n$. Then, for $\lambda \in C_n$, Cauchy's integral formula gives

$$\begin{aligned} m'(\lambda) &= \frac{\partial_\lambda v(2\pi;\lambda)\tilde{v}(2\pi;\lambda) - v(2\pi;\lambda)\partial_\lambda \tilde{v}(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)^2} \\ &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{m(z)}{(z-\lambda)^2}\, dz = -\frac{q(0)}{2\lambda^2} + o\left(\frac{1}{\lambda^{5/2}}\right), \qquad \lambda \to \infty, \quad \lambda \in \bigcup_{n=1}^{\infty} C_n. \end{aligned}$$

Finally, since

$$\frac{\partial_\lambda v(2\pi;\lambda)}{v(2\pi;\lambda)} - \frac{\partial_\lambda \tilde{v}(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)} = \frac{\tilde{v}(2\pi;\lambda)}{v(2\pi;\lambda)} \cdot \frac{\partial_\lambda v(2\pi;\lambda)\tilde{v}(2\pi;\lambda) - v(2\pi;\lambda)\partial_\lambda \tilde{v}(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)^2},$$

we get from the previous asymptotic formulas that

$$\lambda\left[\frac{\partial_\lambda v(2\pi;\lambda)}{v(2\pi;\lambda)}-\frac{\partial_\lambda \tilde{v}(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)}\right]=-\frac{q(0)}{2\lambda}+o\left(\frac{1}{\lambda^{3/2}}\right),\qquad \lambda\to\infty,\quad \lambda\in\bigcup_{n=1}^{\infty}C_n.$$

hence

$$\frac{1}{2\pi i}\oint_{C_n}\lambda\left[\frac{\partial_\lambda v(2\pi;\lambda)}{v(2\pi;\lambda)}-\frac{\partial_\lambda \tilde{v}(2\pi;\lambda)}{\tilde{v}(2\pi;\lambda)}\right]d\lambda=-\frac{q(0)}{2}+o\,(1),\qquad n\to\infty.$$

12 The System of Equations for the μ's

Suppose $q(x)$ is a real C^3 potential and $\mu_1(0), \mu_2(0), \ldots$ are the zeros of $v(2\pi; \lambda)$ associated to $q(x)$. Then (Trubowitz) the system of equations

$$\frac{d\mu_n}{d\xi} = \frac{n^2\sqrt{\Delta(\mu_n)^2 - 4}}{4\prod_{j\neq n}\left(\frac{\mu_j - \mu_n}{j^2/4}\right)}, \qquad n = 1, 2, \ldots,$$

where $\Delta(\lambda)$ is the Hill discriminant associated to $q(x)$, has a unique solution $\mu_1(\xi), \mu_2(\xi), \ldots$.

Under the appropriate choice of the signs of the square roots $\sqrt{\Delta(\mu_n)^2 - 4}$, the solution $\mu_1(\xi), \mu_2(\xi), \ldots$ of the system is the set of zeros of $v_\xi(2\pi; \lambda)$, where $v_\xi(x; \lambda)$ is the solution associated to $q_\xi(x) = q(x + \xi)$ satisfying $v_\xi(0; \lambda) = 0$ and $v'_\xi(0; \lambda) = 1$.

Due to the analytic nature of the above system of equations, we expect that it continues to hold in the case of a smooth complex $q(x)$.

13 A Simple Case where the Conjecture is True

As we have seen, if $\sigma(H) = [0, \infty)$, then $\Delta(\lambda) = 2\cos\left(2\pi\sqrt{\lambda}\right)$, hence $\Delta(\lambda)^2 - 4 = -4\sin^2\left(2\pi\sqrt{\lambda}\right)$. Therefore, the system of equations for the μ's becomes

$$\frac{d\mu_n}{d\xi} = \sigma_n \frac{in^2 \sin\left(2\pi\sqrt{\mu_n}\right)}{2\prod_{j\neq n}\left(\frac{\mu_j - \mu_n}{j^2/4}\right)}, \qquad n = 1, 2, \ldots,$$

where $\sigma_n = \pm 1$.

Suppose now that we have coexistence for all $\lambda \neq 0, m^2/4$, where $m > 0$ is an integer, while for $\lambda = m^2/4$ we do not have coexistence. Then, $v_\xi(2\pi; n^2/4) = 0$ for all $n \geq 1$, $n \neq m$. Consequently, $\mu_n(\xi) = n^2/4$ for all $n \geq 1$, $n \neq m$ and the above system reduces to a single differential equation for $\mu_m(\xi)$:

$$\frac{d\mu_m}{d\xi} = \pm 4i\pi\sqrt{\mu_m}\left(\frac{m^2}{4} - \mu_m\right).$$

This equation can be easily solved and from its solutions we can obtain the associated potentials $q(x)$ (via the trace formula), which turn out to be the Gasymov potentials (recall our Example)

$$q_m(x) = \frac{2m^2 a e^{imx}}{(ae^{imx}+1)^2} = \frac{2m^2 a^{-1} e^{-imx}}{(a^{-1}e^{-imx}+1)^2}, \qquad a \neq 0, \quad |a| \neq 1.$$

Hence our conjecture is true in the case where we have coexistence for all $\lambda \neq 0, m^2/4$.

14 Another Analog of Borg's Theorem

Theorem. Suppose $q \in C^2$ and $\sigma(H) = [\,0, \infty)$, so that $\Delta(\lambda) = 2\cos(2\pi\sqrt{\lambda})$. Furthermore, suppose that we have coexistence at $\lambda = n^2/4$, for every integer $n \geq 1$. Then $q(x) \equiv 0$.

Notice that coexistence at $\lambda = n^2/4$, for every integer $n \geq 1$, implies that both $u(x; n^2/4)$ and $v(x; n^2/4)$ are Floquet solutions and, consequently, periodic or antiperiodic, since $\rho(n^2/4) = \pm 1$. Therefore, $v(2\pi; n^2/4) = 0$ for every integer $n \geq 1$. From the asymptotic formulas it follows that these are the only zeros of $v(2\pi; \lambda)$ and, furthermore, that all these zeros are simple, i.e.

$$\mu_n = \frac{n^2}{4}, \qquad n \geq 1.$$

Furthermore, the same is true for the shifted operator H_ξ, for every $\xi \in \mathbb{R}$.

Therefore by the trace formula we get

$$0 = \sum_{n=1}^{\infty} \left(\frac{n^2}{4} - \frac{n^2}{4} \right) = \sum_{n=1}^{\infty} \left[\mu_n(\xi) - \frac{n^2}{4} \right] = -\frac{q_\xi(0)}{2} \equiv -\frac{q(\xi)}{2},$$

i.e. $q(\xi) \equiv 0$.

15 The Discrete Case

As we all know, discrete and continuous problems share many similarities, but they can also be quite different in certain aspects.

We will briefly mention some results regarding the discrete case. The details can be found in the article:

V.P., Periodic Jacobi operators with complex coefficients, *Journal of Spectral Theory*, **11** (no. 2), 781–819 (2021).

Consider the periodic Jacobi (or discrete Hill-type operator) L defined as

$$(Lw)(n) := a(n)\, w(n+1) + a(n-1)\, w(n-1) + b(n)\, w(n), \qquad n \in \mathbb{Z},$$

where the coefficients $a(n)$ and $b(n)$ are complex-valued and periodic functions of period $N \in \mathbb{N} := \{1, 2, \ldots\}$ with

$$a(n) \neq 0 \qquad \text{for all } \ n \in \mathbb{Z}.$$

Let us mention that if $a(n_0) = 0$ for some n_0, then $a(n_0 + \ell N) = 0$ for all $\ell \in \mathbb{Z}$ and L splits as $L = \bigoplus_{\ell \in \mathbb{Z}} A$, where A is a linear operator acting on an N-dimensional space, which can be considered as a degenerate case (e.g., the spectrum of L consists of at most N eigenvalues of infinite multiplicity).

In the special case where $a(n) \equiv -1$ the operator L becomes the **one-dimensional discrete periodic Schrödinger** (or **discrete Hill**) operator with potential $b(n)$.

Unlike the continuous case, here there is no Liouville-type transformation which transforms the general operator L to a discrete periodic Schrödinger operator (in fact, even in the continuous case, in the presence of complex coefficients the Liouville transformation becomes problematic).

We will normalize (unless otherwise stated) $a(n)$ so that

$$\prod_{j=1}^{N} a(j) = (-1)^N$$

Since $a(n)$ and $b(n)$ are N-periodic, they can be expanded as

$$a(n) = A_0 + \sum_{k=1}^{N-1} A_k \omega_N^{kn} \quad \text{and} \quad b(n) = B_0 + \sum_{k=1}^{N-1} B_k \omega_N^{kn}, \qquad \omega_N := e^{2\pi i/N},$$

where $A_0, A_1, \ldots, A_{N-1}, B_0, B_1, \ldots, B_{N-1} \in \mathbb{C}$ (this is a Fourier-style expansion).

It seems there is no analog of a Gasymov potential in the discrete case.

It is not hard to check the orthogonality relation

$$\sum_{n=0}^{N-1} \omega_N^{jn}\,\bar{\omega}_N^{kn} = \sum_{n=0}^{N-1} e^{2(j-k)n\pi i/N} = N\delta_{jk} \qquad \text{for } \; j,k = 0,1,\ldots,N-1,$$

where the bar denotes complex conjugation (thus $\bar{\omega}_N = \omega_N^{-1}$) and δ_{jk} is the Kronecker delta. We have

$$A_k = \frac{1}{N}\sum_{n=0}^{N-1} a(n)\,\bar{\omega}_N^{kn} \qquad \text{and} \qquad B_k = \frac{1}{N}\sum_{n=0}^{N-1} b(n)\,\bar{\omega}_N^{kn},$$

in particular (for $k = 0$),

$$\sum_{n=0}^{N-1} a(n) = A_0 N \qquad \text{and} \qquad \sum_{n=0}^{N-1} b(n) = B_0 N.$$

Clearly our discrete operator L is bounded on $l^2(\mathbb{Z})$, and hence the $l^2(\mathbb{Z})$-spectrum $\sigma(L)$ of L is a compact subset of $\mathbb{C}$.

16 Review of the Spectral Theory of L

The spectral theory of L is studied via the equation

$$(Lw)(n) = a(n)\,w(n+1) + a(n-1)\,w(n-1) + b(n)\,w(n) = \lambda w(n), \qquad n \in \mathbb{Z},$$

where $\lambda \in \mathbb{C}$ is the spectral parameter.

It is customary to introduce the two solutions $u(n) = u(n;\lambda)$ and $v(n) = v(n;\lambda)$ which satisfy the initial conditions

$$u(-1) = 0, \quad u(0) = 1, \qquad\qquad v(-1) = -\frac{1}{a(-1)}, \quad v(0) = 0.$$

For $n \geq 0$ the solution $u(n) = u(n; \lambda)$ is a polynomial in λ of degree n having the form

$u(n; \lambda) =$

$$\frac{1}{\Pi_n} \left\{ \lambda^n - \left[\sum_{j=0}^{n-1} b(j) \right] \lambda^{n-1} + \left[\sum_{0 \leq j < k \leq n-1} b(j)\, b(k) - \sum_{j=0}^{n-2} a(j)^2 \right] \lambda^{n-2} + \cdots \right\},$$

where

$$\Pi_n := \prod_{j=0}^{n-1} a(j),$$

while for $n \geq 1$ the solution $v(n) = v(n; \lambda)$ is a polynomial in λ of degree $n - 1$ having the form

$v(n; \lambda) =$

$$\frac{1}{\Pi_n} \left\{ \lambda^{n-1} - \sum_{j=1}^{n-1} b(j) \lambda^{n-2} + \left[\sum_{1 \leq j < k \leq n-1} b(j)\, b(k) - \sum_{j=1}^{n-2} a(j)^2 \right] \lambda^{n-3} + \cdots \right\}$$

(here we follow the standard convention that empty sums equal 0, while empty products equal 1; e.g., $v(1; \lambda) = a(0)^{-1}$).

Notice also that

$$\begin{vmatrix} u(n) & v(n) \\ -a(n-1)\,u(n-1) & -a(n-1)\,v(n-1) \end{vmatrix} = 1 \qquad \text{for all } \; n \in \mathbb{Z}, \; \lambda \in \mathbb{C}.$$

In particular, $u(n)$ and $v(n)$ are linearly independent solutions of (1) for any value of the parameter λ.

Sometimes it is more convenient, instead of the solutions $u(n)$ and $v(n)$ to work with the (linearly independent) solutions $\chi(n) = \chi(n;\lambda)$ and $\gamma(n) = \gamma(n;\lambda)$ determined by the initial conditions

$$\chi(0) = 1, \quad \chi(1) = 0 \qquad \text{and} \qquad \gamma(0) = 0, \quad \gamma(1) = 1.$$

It follows easily that

$$\chi(n;\lambda) = u(n;\lambda) + [b(0) - \lambda]v(n;\lambda) \qquad \text{and} \qquad \gamma(n;\lambda) = a(0)\, v(n;\lambda).$$

For $n \geq 2$, we have that $\chi(n;\lambda)$ is a polynomial in λ of degree $n-2$ and $\gamma(n;\lambda)$ is a polynomial in λ of degree $n-1$. Finally, an easy calculation yields

$$\begin{vmatrix} \chi(n) & \gamma(n) \\ \chi(n+1) & \gamma(n+1) \end{vmatrix} = \frac{a(0)}{a(n)} \qquad \text{for all } \ n \in \mathbb{Z}, \ \lambda \in \mathbb{C}.$$

Remark. For any fixed $n \geq 0$ the polynomials $v(n;\lambda)$ and $v(n+1;\lambda)$ do not have any common zeros (i.e. they are relatively prime). The justification of this fact is very simple: Suppose $v(n;\lambda_0) = v(n+1;\lambda_0) = 0$. Then, the fact that $v(n;\lambda_0)$ satisfies the difference equation (for $\lambda = \lambda_0$) implies that $v(n;\lambda_0) = 0$ for all $n \in \mathbb{Z}$, which is a contradiction since, e.g., $v(1;\lambda_0) = 1/a(0)$. Likewise, the polynomials $u(n;\lambda)$ and $u(n+1;\lambda)$ do not share any common zeros for any fixed $n \geq 0$ and the same is true for $\chi(n;\lambda)$ and $\chi(n+1;\lambda)$ as well as for $\gamma(n;\lambda)$ and $\gamma(n+1;\lambda)$. $\diamondsuit$

Now let $\mathcal{S}$ be the "N-shift" operator

$$(\mathcal{S}f)(n) := f(n+N).$$

Our assumption $a(n+N) = a(n)$ and $b(n+N) = b(n)$ for all $n \in \mathbb{Z}$ implies that the linear operator $\mathcal{S}$ maps solutions to solutions of for the same value of λ (in other words, $\mathcal{S}$ commutes with L), and by exploiting this very simple property we can derive the (Floquet) spectral theory of L.

For each $\lambda \in \mathbb{C}$ let $\mathcal{W} = \mathcal{W}(\lambda)$ be the two-dimensional vector space of the solutions. By the previous discussion, for each $\lambda \in \mathbb{C}$ the solutions u and v of can be taken as a basis of $\mathcal{W}(\lambda)$, and the matrix of the operator $S|_{\mathcal{W}}$ with respect to the basis (u, v) is

$$S = S(\lambda) = \begin{bmatrix} u(N;\lambda) & v(N;\lambda) \\ -a(-1)\,u(N-1;\lambda) & -a(-1)\,v(N-1;\lambda) \end{bmatrix}$$

(we should not forget that the matrix S and the vector space $\mathcal{W}$ depend on λ). Thus, S is the Floquet (or monodromy) matrix, and the fact that $a(n)$ is N-periodic yields

$$\det S(\lambda) \equiv 1.$$

It follows that the characteristic polynomial of $S(\lambda)$ has the form

$\det(S - \rho I) = \rho^2 - \Delta(\lambda)\,\rho + 1,$ where $\Delta(\lambda) := \mathrm{tr} S = u(N;\lambda) - a(-1)\,v(N-1;\lambda).$

Also, it is easy to check that $\Delta(\lambda)$ can be expressed in terms of the solutions $\chi(n;\lambda)$ and $\gamma(n;\lambda)$ as

$$\Delta(\lambda) = \chi(N;\lambda) + \gamma(N+1;\lambda).$$

The quantity $\Delta(\lambda)$ is the (discrete) Hill discriminant of L and it follows that it is a polynomial of λ of degree N having the form

$$\Delta(\lambda) =$$

$$(-1)^N \left\{ \lambda^N - B_0 N \lambda^{N-1} + \left[\sum_{1 \le j < k \le N} b(j)\, b(k) - \sum_{j=1}^{N} a(j)^2 \right] \lambda^{N-2} + \cdots \right\}.$$

The eigenvalues $\rho_1(\lambda)$ and $\rho_2(\lambda)$ of S are the Floquet multipliers, while their corresponding eigenvectors $\phi_1(n;\lambda)$ and $\phi_2(n;\lambda)$ are the Floquet solutions so that

$$\phi_j(n+N) = (\mathcal{S}\phi_j)(n) = \rho_j \phi_j(n), \qquad j = 1, 2.$$

We have

$$\rho_1(\lambda)\,\rho_2(\lambda) \equiv 1 \qquad \text{and} \qquad \rho_1(\lambda) + \rho_2(\lambda) = \Delta(\lambda),$$

so that

$$\rho_1(\lambda), \rho_2(\lambda) = \frac{\Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4}}{2}.$$

Let us also notice that $S(\lambda)$ can have a Jordan anomaly only if $\rho_1(\lambda) = \rho_2(\lambda) = \pm 1$ (equivalently, only if $\Delta(\lambda) = \pm 2$) and in the presence of such an anomaly the matrix $S(\lambda)$ is similar to the canonical matrix

$$\begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}.$$

If this is the case, then there is only one Floquet solution $\phi(n)$ satisfying $\phi(n+N) = \pm\phi(n)$, while there is a second solution $g(n)$ (sometimes called a generalized Floquet solution), linearly independent to $\phi(n)$, satisfying

$$g(n+N) = \pm g(n) + \phi(n) \qquad \text{for all } \ n \in \mathbb{Z}.$$

Recall, however, that even if $\rho_1(\lambda) = \rho_2(\lambda) = \pm 1$, the Floquet matrix may still be diagonalizable (and, hence, $S(\lambda) = \pm I$, where I is the 2×2 identity matrix), in which case we have coexistence of two periodic (if $S(\lambda) = I$) or antiperiodic (if $S(\lambda) = -I$), linearly independent Floquet solutions.

Remark. Suppose that a function $\phi(x)$ satisfies

$$\phi(x+N) = \rho\phi(x) \qquad \text{for all } \; x \in \mathbb{R},$$

where $\rho \neq 0$ is a constant. We write

$$\rho = e^{\beta N}$$

and set

$$p(x) := e^{-\beta x}\phi(x).$$

Then, $p(x)$ is N-periodic and $\phi(x)$ can be written as

$$\phi(x) = e^{\beta x}p(x), \qquad \text{where } \; p(x+N) = p(x).$$

Suppose now that $g(x)$ satisfies

$$g(x+N) = \rho g(x) + \phi(x) \qquad \text{for all } \; x \in \mathbb{R}.$$

We set

$$p_1(x) := e^{-\beta x}g(x) - \frac{x}{N\rho}\,p(x).$$

Then,

$$p_1(x+N) = e^{-\beta x}e^{-\beta N}[\rho g(x) + \phi(x)] - \frac{x}{N\rho}\,p(x) - \frac{1}{\rho}\,p(x) = p_1(x).$$

Therefore, $g(x)$ can be expressed as

$$g(x) = e^{\beta x}p_1(x) + \frac{x}{N\rho}\,e^{\beta x}p(x) = e^{\beta x}p_1(x) + \frac{x}{N\rho}\,\phi(x),$$

where $p_1(x)$ and $p(x)$ are N-periodic.
Finally, let us mention that all the above are valid if the functions are defined only for $x \in \mathbb{Z}$, provided, of course, that $N \in \mathbb{Z}$. $\diamondsuit$

It is sometimes more convenient to view $\rho_1(\lambda)$ and $\rho_2(\lambda)$ as the two branches of a (single-valued) analytic function $\rho(\lambda)$ defined on the Riemann surface Σ of the function $\sqrt{\Delta(\lambda)^2 - 4}$. Then,

$$\rho(\lambda) = \frac{\Delta(\lambda) + \sqrt{\Delta(\lambda)^2 - 4}}{2}$$

and $\rho(\lambda)$ can be called the Floquet multiplier associated to L. Let us notice that, since $\Delta(\lambda)^2 - 4$ is a polynomial of even degree, Σ has two points at ∞. If Σ_{fin} denotes the set of finite points of Σ (namely Σ minus its two points at ∞), then $\rho(\lambda)$ has neither zeros nor poles in Σ_{fin}. As for the two points at ∞ of Σ, since $\Delta(\lambda)$ has degree N it follows that at one of these points $\rho(\lambda)$ has a zero of multiplicity N, while at the other it has a pole of order N. Also,

$$\Delta(\lambda) = \rho(\lambda) + \frac{1}{\rho(\lambda)}$$

and (ii) that

$$\frac{\rho'(\lambda)}{\rho(\lambda)} = \frac{\Delta'(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}.$$

The Floquet solutions too can be viewed as the two branches of a meromorphic function defined on Σ. First we normalize them so that $\phi_1(0;\lambda) = \phi_2(0;\lambda) = 1$. It, then, follows that $\phi_1(n;\lambda)$ and $\phi_2(n;\lambda)$ are the branches of the function

$$\phi(n;\lambda) = u(n;\lambda) - \frac{u(N;\lambda) - \rho(\lambda)}{v(N;\lambda)}\, v(n;\lambda).$$

As we have already mention, $\deg_\lambda v(N;\lambda) = N-1$ (where $\deg_\lambda v(N;\lambda)$ denotes the degree of $v(N;\lambda)$ viewed as a polynomial of λ). Hence, $\phi(n;\lambda)$, as a function of λ, can have at most $N-1$ poles in Σ_{fin} counting multiplicities (in the non-self-adjoint case the zeros of $v(N;\lambda)$ are not necessarily simple — we will see an example where $v(4;\lambda)$ has a triple zero).

Having $\Delta(\lambda)$ and $r(\lambda)$, the spectrum $\sigma(L)$ of L can be characterized as

$$\sigma(L) = \{\lambda \in \mathbb{C} \,:\, \Delta(\lambda) \in [-2, 2\,]\} \quad \Leftrightarrow \quad \sigma(L) = \{\lambda \in \mathbb{C} \,:\, |\rho(\lambda)| = 1\},$$

which implies that $\sigma(L)$ is a finite union of bounded analytic arcs lying in the complex plane (notice that $|\rho_1(\lambda)| = 1$ if and only if $|\rho_2(\lambda)| = 1$).

The adjoint operator L^* of L is given by the formula

$$(L^* w)(x) = \overline{a(n)}\, w(n+1) + \overline{a(n-1)}\, w(n-1) + \overline{b(n)}\, w(n), \qquad n \in \mathbb{Z},$$

where the bar denotes complex conjugation. Hence L is self-adjoint if and only if $a(n)$ and $b(n)$ are real-valued. In general we have

$$\sigma(L^*) = \overline{\sigma(L)},$$

namely $\lambda \in \sigma(L^*)$ if and only if $\overline{\lambda} \in \sigma(L)$. If in particular $\sigma(L) \subset \mathbb{R}$, then $\sigma(L^*) = \sigma(L)$.

16.1 Floquet Spectrum; Periodic and Antiperiodic Eigenvalues

Other ways to express the spectrum are

$$\sigma(L) = \{\lambda \in \mathbb{C} \,:\, \Delta(\lambda) = 2\cos(\kappa), \quad 0 \le \kappa \le \pi\}$$

and

$$\sigma(L) = \{\lambda \in \mathbb{C} \,:\, r_1(\lambda) = e^{i\kappa}, \quad 0 \le \kappa \le \pi\}.$$

Thus, if for a given $\kappa \in [0, \pi]$, we introduce the *Floquet spectrum*

$$\sigma_\kappa(L) := \{\lambda \in \mathbb{C} \,:\, \Delta(\lambda) = 2\cos(\kappa)\},$$

then $\sigma(L)$ can be written as the disjoint union

$$\sigma(L) = \bigcup_{0 \le \kappa \le \pi} \sigma_\kappa(L).$$

Clearly, the Floquet spectrum $\sigma_\kappa(L)$ is the set of zeros of the N-th degree polynomial

$$F_\kappa(\lambda) := \Delta(\lambda) - 2\cos(\kappa).$$

Observe that $F_\kappa'(\lambda) = \Delta'(\lambda)$ is independent of κ and has degree $N - 1$. Thus, if λ is a multiple zero of $F_\kappa(\lambda)$, then λ must be a zero of $\Delta'(\lambda)$, and there are at most $N - 1$ such zeros (which, of course, are independent of κ). For each such value of λ there is at most one $\kappa \in [0, \pi]$ for which $F_\kappa(\lambda) = 0$ (since $\cos(\kappa)$ is strictly decreasing on $[0, \pi]$). It follows that there are at most $N - 1$ values of $\kappa \in [0, \pi]$ for which $F_\kappa(\lambda)$ has multiple zeros and, therefore, if κ is not equal to any of those exceptional values, the Floquet spectrum $\sigma_\kappa(L)$ consists of N distinct κ-*Floquet eigenvalues*. In the self-adjoint case, if $\kappa \ne 0, \pi$, then $F_\kappa(\lambda)$ has N distinct zeros.

Let us first consider the case $\kappa \in (0, \pi)$, namely $\kappa \neq 0$ and $\kappa \neq \pi$. Under this assumption for κ, if $\lambda \in \sigma_\kappa(L)$, we have $r_1(\lambda) = e^{i\kappa} \neq \pm 1$ and, therefore, λ is not a branch point of $r(\lambda)$, hence there are two linearly independent Floquet solutions $\phi_1(n) = \phi_1(n; \lambda)$ and $\phi_1(n) = \phi_2(n; \lambda)$ corresponding to any particular $\lambda \in \sigma_\kappa(L)$, satisfying

$$\phi_1(n + N) = e^{i\kappa}\phi_1(n), \qquad n \in \mathbb{Z}$$

and

$$\phi_2(n + N) = e^{-i\kappa}\phi_2(n), \qquad n \in \mathbb{Z}.$$

Then,

$$\phi_1(n) = e^{i\kappa n/N}p(n), \qquad \text{where } \; p(n + N) = p(n).$$

Notice that $p(n)$ satisfies the boundary value problem

$$a(n)e^{i\kappa/N}p(n + 1) + a(n - 1)e^{-i\kappa/N}p(n - 1) + b(n)\,p(n) = \lambda p(n),$$

$$p(0) = p(N), \qquad\qquad p(1) = p(N + 1).$$

The problem can be written in the matrix form

$$M_\kappa \vec{p} = \lambda \vec{p}$$

where $\vec{p}$ is the column vector $\vec{p} := [p(0), \ldots, p(N-1)]^\top$ and M_κ is the $N \times N$ matrix (for $N \geq 3$)

$$M_\kappa := \begin{bmatrix} b(0) & a(0)e^{i\kappa/N} & 0 & \cdots & a(N-1)e^{-i\kappa/N} \\ a(0)e^{-i\kappa/N} & b(1) & a(1)e^{i\kappa/N} & \cdots & 0 \\ 0 & a(1)e^{-i\kappa/N} & b(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a(N-2)e^{i\kappa/N} \\ a(N-1)e^{i\kappa/N} & 0 & 0 & \cdots & b(N-1) \end{bmatrix}.$$

If κ is such that the polynomial $F_\kappa(\lambda)$ has simple zeros, then $F_\kappa(\lambda)$ must be the characteristic polynomial of M_κ. Then, by continuity we have that

$$\det(M_\kappa - \lambda I) = \Delta(\lambda) - 2\cos(\kappa) \qquad \text{for all} \;\; \kappa \in [0, \pi]$$

(and, consequently, by analytic continuation the above equation must hold for all $\kappa \in \mathbb{C}$). In particular, the spectrum of M_κ is $\sigma_\kappa(L)$. Furthermore, the (pure) eigenvectors of M_κ correspond precisely to the Floquet solutions. Let us recall that in the self-adjoint case λ is a branch point of $r(\lambda)$ if and only if $S(\lambda)$ has a Jordan anomaly (and such a λ must necessarily be real). However, this is not always true in the non-self-adjoint case.

For an integer $m \geq 1$ let us consider the space

$$\mathcal{P}_m := \{f(n) : f(n+m) = f(n) \quad \text{for all} \;\; n \in \mathbb{Z}\},$$

namely the set of m-periodic sequences over the complex numbers. Obviously, $\mathcal{P}_m$ is a vector space of (complex) dimension m. In the case where m is a multiple of N, the operator L, having N-periodic coefficients $a(n)$ and $b(n)$, maps $\mathcal{P}_m$ into $\mathcal{P}_m$. In particular, for $m = 2N$ the operator L maps $\mathcal{P}_{2N}$ into $\mathcal{P}_{2N}$. As a basis of $\mathcal{P}_{2N}$ we can choose the sequences

$$e_j(n) := \delta_{jn}, \qquad n \in \mathbb{Z}, \;\; j = 1, \ldots, 2N,$$

where δ_{jn} is the Kronecker delta. Then, the $2N \times 2N$ matrix of $L|_{\mathcal{P}_{2N}}$ with respect to that basis is

$$
L_{2N} := \begin{bmatrix} b(1) & a(1) & 0 & \cdots & 0 & 0 & a(2N) \\ a(1) & b(2) & a(2) & \cdots & 0 & 0 & 0 \\ 0 & a(2) & b(3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b(2N-2) & a(2N-2) & 0 \\ 0 & 0 & 0 & \cdots & a(2N-2) & b(2N-1) & a(2N-1) \\ a(2N) & 0 & 0 & \cdots & 0 & a(2N-1) & b(2N) \end{bmatrix},
$$

where $a(n+N) = a(n)$ and $b(n+N) = b(n)$. Notice that the matrix L_{2N} is symmetric, but not Hermitian, unless, of course $a(n)$ and $b(n)$ are real-valued, in which case L_{2N} is real symmetric (hence Hermitian) and its associated operator $L|_{\mathcal{P}_{2N}}$ is self-adjoint.

It follows that the eigenvectors of $L|_{\mathcal{P}_{2N}}$ (in $\mathcal{P}_{2N}$), being $2N$-periodic solutions, are precisely the N-periodic and N-antiperiodic (linearly independent) solutions. Also, the spectrum of the operator $L|_{\mathcal{P}_{2N}}$, i.e. the set of eigenvalues of the matrix L_{2N}, coincides with the set of periodic and antiperiodic eigenvalues of L, that is the zeros of the polynomials $\Delta(\lambda)-2$ and $\Delta(\lambda)+2$. We know that $\Delta(\lambda)$ has degree N. Hence the polynomial $\Delta(\lambda)-2$ has at most N distinct zeros and the same is true for $\Delta(\lambda)+2$. Obviously, these two polynomials cannot have common zeros. On the other hand $\Delta(\lambda)-2$ and $\Delta(\lambda)+2$ have the same derivative, namely $\Delta'(\lambda)$, from which it follows that $\Delta(\lambda)^2-4$ has at least $N+1$ distinct zeros (the derivative of $\Delta(\lambda)^2-4$ is $2\Delta(\lambda)\Delta'(\lambda)$ and $\Delta(\lambda)$ does not have common zeros with $\Delta(\lambda)^2-4$).

$\Delta(\lambda)^2-4$ is a monic polynomial of degree $2N$, i.e. its leading term is λ^{2N}. Also, for generic $a(n)$ and $b(n)$ we have that $\Delta(\lambda)^2-4$ has $2N$ simple zeros. Therefore, the characteristic polynomial of L_{2N} is

$$
\det(L_{2N} - \lambda I) = \Delta(\lambda)^2 - 4.
$$

16.2 Certain Classes of Isospectral Operators

Let $a^l(n) := a(n+l)$ and $b^l(n) := b(n+l)$, where $l \in \mathbb{Z}$, and consider the operator

$$(L^l w)(n) := a^l(n)\, w(n+1) + a^l(n-1)\, w(n-1) + b^l(n)\, w(n), \qquad n \in \mathbb{Z}.$$

Since

$$\phi_j(n+l+N;\lambda) = r_j(\lambda)\phi_j(n+l;\lambda), \qquad j = 1, 2.$$

we can see that L and L^l have the same multiplier $r(\lambda)$ and, consequently, the same discriminant. Hence, $\sigma(L^l) = \sigma(L)$.

Also, if $a^\sharp(n) := a(-n)$, $b^\sharp(n) := b(-n)$, and $L^\sharp$ is the operator associated to $a^\sharp$ and $b^\sharp$, then $\phi(-n;\lambda)$ is a Floquet solution of $L^\sharp w = \lambda w$ if and only if $\phi(n;\lambda)$ is a Floquet solution of $Lw = \lambda w$. It follows that L and $L^\sharp$ have the same multiplier, the same discriminant, and the same spectrum.

Proposition 1. Suppose $\tau(n) = 1$ or -1 and $\tau(n+N) = \tau(n)$ for all $n \in \mathbb{Z}$. Let $a(n)$ and $b(n)$ be the coefficients of the (N-periodic) Jacobi operator L and consider the operator $\hat{L}$ whose coefficients are $\hat{a}(n) := \tau(n)a(n)$ and $\hat{b}(n) := b(n)$. If $\Delta(\lambda)$ and $\hat{\Delta}(\lambda)$ are the discriminants of L and $\hat{L}$ respectively, then

$$\hat{\Delta}(\lambda) = (-1)^{\nu(\tau)}\Delta(\lambda), \qquad \text{where } \nu(\tau) := \#\{n : \tau(n) = -1,\ 0 \le n \le N-1\}$$

($\#\mathcal{S}$ denotes the cardinality of the set $\mathcal{S}$). In particular, $\sigma(\hat{L}) = \sigma(L)$.

Motivated by Proposition 1 we introduce the following equivalence relation between Jacobi operators.

Definition. Two Jacobi operators L and $\hat{L}$ (of complex coefficients) are called equivalent, symbolically $L \sim \hat{L}$, if their associated coefficients $a(n)$, $b(n)$, $\hat{a}(n)$, and $\hat{b}(n)$ are related as $\hat{a}(n) = \tau(n)a(n)$ and $\hat{b}(n) = b(n)$ for all $n \in \mathbb{Z}$, where $\tau(n) = 1$ or -1.

In other words, $L \sim \hat{L}$, if $a(n)^2 = \hat{a}(n)^2$ and $\hat{b}(n) = b(n)$ for all $n \in \mathbb{Z}$.

Remark. Clearly, in the N-periodic case, if the coefficient $a(n)$ of L satisfies the normalization, then the coefficient $\hat{a}(n) = \tau(n)a(n)$ of $\hat{L}$ satisfies it if and only if $\nu(\tau)$ satisfies $(-1)^{\nu(\tau)} = 1$ (i.e. $\nu(\tau)$ is even). Thus, if the coefficients of both L and $\hat{L}$ satisfy the normalization and $L \sim \hat{L}$, then $\hat{\Delta}(\lambda) = \Delta(\lambda)$. $\diamondsuit$

16.3 The Dirichlet Spectrum

Let us look at the Dirichlet-type boundary value problem ($N \geq 2$)

$$(L\psi)(n) = a(n)\,\psi(n+1) + a(n-1)\,\psi(n-1) + b(n)\,\psi(n) = \mu\,\psi(n) \tag{2}$$
$$\psi(0) = \psi(N) = 0 \tag{3}$$

(notice that $\psi(n)$ can be extended for all $n \in \mathbb{Z}$). Clearly, the eigenvalues of the problem are the zeros of the polynomial $v(N;\lambda)$. As we have seen $\deg_\lambda v(N;\lambda) = N-1$, hence there are $N-1$ Dirichlet eigenvalues $\mu_1, \ldots, \mu_{N-1}$, counting multiplicities. Hence,

$$v(N;\lambda) = (-1)^N \prod_{j=1}^{N-1} (\lambda - \mu_j) \qquad \text{and} \qquad \gamma(N;\lambda) = (-1)^N a(0) \prod_{j=1}^{N-1} (\lambda - \mu_j)$$

In the case where $a(n)$ and $b(n)$ are real-valued, the problem (2)–(3) is self-adjoint and hence the eigenfunctions form a basis of the underlying vector space, which is clearly $(N-1)$-dimensional. Since for each μ_j we cannot have more than one Dirichlet eigenfunction (up to linear independence), it follows that in the real case the zeros of $v(N;\lambda)$ are real and simple (and between any two bands of the spectrum there is exactly one Dirichlet eigenvalue).

However, this is not always true in the case of nonreal $a(n)$, $b(n)$. For example, if $N = 4$ and $a(n) \equiv -1$, then

$$v(4;\lambda) = -[\lambda - b(1)][\lambda - b(2)][\lambda - b(3)]) + \lambda - b(1) + \lambda - b(3),$$

and the choice $b(1) = -b(3) = \sqrt{2}\,i$ and $b(2) = 0$ yields $v(4;\lambda) = -\lambda^3$, hence $\mu_1 = \mu_2 = \mu_3 = 0$. Fixing $b(4) = 0$ gives a specific "pathological" example, namely

$$a(n) \equiv -1, \qquad b(n) = \frac{i^n - (-i)^n}{\sqrt{2}}.$$

We also have the "trace formula"

$$\mu_1 + \cdots + \mu_{N-1} = b(1) + \cdots + b(N-1),$$

which can be also written as

$$\mu_1 + \cdots + \mu_{N-1} = B_0 N - b(0).$$

Also, since

$$\Delta(\lambda)^2 - 4 = \left\{\lambda^{2N} - 2\,[b(1) + \cdots + b(N)]\,\lambda^{2N-1} + \cdots\right\},$$

it follows that

$$\sum_{j=0}^{2N-1} \lambda_j = 2\,[b(1) + \cdots + b(N)] = 2B_0 N,$$

where λ_j, $j = 0, 1, \ldots, 2N-1$, are the zeros of $\Delta(\lambda)^2 - 4$ (counting multiplicities), namely the periodic and antiperiodic eigenvalues. Furthermore,

$$\sum_{j=0}^{2N-1} \lambda_j - 2\sum_{j=1}^{N-1} \mu_j = 2b(0).$$

Finally, let us mention that the Dirichlet eigenfunction $\psi(n)$, extended to $\mathbb{Z}$, is always a Floquet solution.

16.4 The Unperturbed Case

If $a(n) \equiv -1$ and $b(n) \equiv 0$ (viewed as N-periodic functions), then the operator L reduces to the unperturbed operator

$$\left(\tilde{L}w\right)(n) := -w(n+1) - w(n-1), \qquad n \in \mathbb{Z},$$

and equation (1) becomes

$$\left(\tilde{L}w\right)(n) = -w(n+1) - w(n-1) = \lambda w(n), \qquad n \in \mathbb{Z}.$$

A tilded quantity will be always associated with the unperturbed case.
It is convenient to introduce a new spectral parameter z related to λ as

$$z + z^{-1} := -\lambda.$$

The solutions χ and γ in the unperturbed case become

$$\tilde{\chi}(n;\lambda) = \frac{z^{1-n} - z^{n-1}}{z - z^{-1}}, \qquad \text{and} \qquad \tilde{\gamma}(n;\lambda) = \frac{z^{n} - z^{-n}}{z - z^{-1}}.$$

In particular, for $\lambda = -2$ (equivalently $z = 1$) we have

$$\tilde{\chi}(n;-2) = 1 - n \qquad \text{and} \qquad \tilde{\gamma}(n;-2) = n,$$

while for $\lambda = 2$ (equivalently $z = -1$) we have

$$\tilde{\chi}(n;2) = (-1)^{n-1} n \qquad \text{and} \qquad \tilde{\gamma}(n;2) = (-1)^{n-1}(n-1).$$

By straightforward induction we can also see that the solution $\tilde{\gamma}(n;\lambda)$, $n \geq 3$, expanded in descending powers of λ, has the form

$$\tilde{\gamma}(n;\lambda) = (-1)^{n-1}\lambda^{n-1} + (-1)^{n}(n-2)\lambda^{n-3} + \cdots$$

The discriminant of the unperturbed operator is

$$\tilde{\Delta}_N(\lambda) := z^N + z^{-N} = \left(\frac{-\lambda + \sqrt{\lambda^2 - 4}}{2}\right)^N + \left(\frac{-\lambda - \sqrt{\lambda^2 - 4}}{2}\right)^N.$$

Also, from the fact that for $n \geq 2$ we have (in the unperturbed case) that $\tilde{u}(n;\lambda) = -\tilde{v}(n-1;\lambda)$, we obtain the expansion

$$\tilde{\Delta}_N(\lambda) = (-1)^N \lambda^N - (-1)^N N \lambda^{N-2} + \cdots, \qquad \text{for } N \geq 2.$$

Thus, in particular, the coefficient of λ^{N-1} in $\tilde{\Delta}_N(\lambda)$ is 0.

the Floquet multiplier becomes

$$\tilde{\rho}(\lambda) = z^N = \left(\frac{-\lambda + \sqrt{\lambda^2 - 4}}{2}\right)^N,$$

while the spectrum is

$$\sigma\left(\tilde{L}\right) = [-2, 2].$$

Furthermore, if we set

$$z_k := e^{i\pi k/N}, \qquad k = 0, 1, \ldots, 2N - 1,$$

and

$$\tilde{\lambda}_k := -\left(z_k + z_k^{-1}\right) = -2\cos\left(\frac{\pi k}{N}\right), \qquad k = 0, 1, \ldots, 2N - 1,$$

then $\tilde{\lambda}_0 = -2$, $\tilde{\lambda}_N = 2$ and $\tilde{\lambda}_k = \tilde{\lambda}_{2N-k}$ for $k = 1, \ldots, N-1$. In addition, $\tilde{\lambda}_0$ and $\tilde{\lambda}_N$ are simple zeros of $\tilde{\Delta}_N(\lambda)^2 - 4$, while $\tilde{\lambda}_k$, $k = 1, \ldots, N-1$, are double zeros of $\tilde{\Delta}_N(\lambda)^2 - 4$. It follows that $\tilde{\lambda}_0 = -2$ is a periodic eigenvalue of $\tilde{L}$ of geometric multiplicity 1, the corresponding eigenfunction being $\tilde{\phi}(n; -2) \equiv 1$, while $\tilde{\lambda}_N = 2$ is a periodic (antiperiodic) eigenvalue of $\tilde{L}$ of geometric multiplicity 1, if N is even (odd), the corresponding eigenfunction being $\tilde{\phi}(n; 2) = (-1)^n$. Finally, each $\tilde{\lambda}_k = 2\cos(\pi k/N)$, $k = 1, \ldots, N-1$, is a periodic (antiperiodic) eigenvalue of $\tilde{L}$ of geometric multiplicity 2, if k is even (odd), while the associated eigenfunctions are $\tilde{\phi}_1(n; \tilde{\lambda}_k) = e^{i\pi kn/N}$ and $\tilde{\phi}(n; \tilde{\lambda}_k) = e^{-i\pi kn/N}$ (i.e. we have coexistence of two linearly independent periodic or antiperiodic solutions).

16.5 The Essentially Unperturbed Operators

Definition. We say that L is an *essentially unperturbed operator* if $L \sim \tilde{L}$, i.e. if $a(n)^2 \equiv 1$ and $b(n) \equiv 0$.

From the above definition it follows that, for a given period N there are 2^N essentially unperturbed operators, one of them being $\tilde{L}$. Obviously the essentially unperturbed operators have real coefficients and hence they are self-adjoint. Notice also that L is essentially unperturbed if and only if $-L$ is essentially unperturbed. If N is odd and L is essentially unperturbed, then either L or $-L$ satisfies the normalization.

Remark There are many results which can be proved by first checking that they are valid for the essentially unperturbed case and then view the general case as a continuous deformation of the unperturbed case. For instance, let us show that for any real $a(n)$ and $b(n)$ (with $a(n) \neq 0$ for all n) the zeros of the polynomials $v(N;\lambda)$ and $v(N+1;\lambda)$ interlace. In the case of $a(n) \equiv \pm 1$ and $b(n) \equiv 0$ the statement follows easily. Now, given any $a(n) \neq 0$ and $b(n)$ consider the family of quantities $a(n;t) \neq 0$ and $b(n;t)$, $t \in [0,1]$ continuous in t, such that $a(n;0) = \mathrm{sgn}[a(n)]$, $a(n;1) = a(n)$, $b(n;0) = 0$, and $b(n;1) = b(n)$ (e.g., $b(n;t) = tb(n)$). For each t the zeros of $v(N;\lambda;t)$ and $v(N+1;\lambda;t)$ (where $v(n;\lambda;t)$ denotes the solution when the coefficients of L are $a(n;t)$ and $b(n;t)$), such that $v(0;\lambda;t) = 0$ and $v(1;\lambda;t) = 1$) are real, being the Dirichlet eigenvalues of a self-adjoint operator. Furthermore, as t moves continuously from 0 to 1 no zero of $v(N;\lambda;t)$ can "cross" a zero of $v(N+1;\lambda;t)$ due to our Remark. Hence, the relative position of the zeros of $v(N;\lambda;t)$ and $v(N+1;\lambda;t)$ is independent of t. Since for $t = 0$ their zeros interlace, it follows that they also interlace for $t = 1$. $\diamondsuit$

17 Inverse Spectral Considerations for the Case of a Discrete Schrödinger Operator

$$(L_{\text{Schr}}\, w)(n) := -w(n+1) - w(n-1) + b(n)\, w(n), \qquad n \in \mathbb{Z}.$$

Proposition 2. In the case $L = L_{\text{Schr}}$ the zeros of $v(N;\lambda)$ (counting multiplicities) together with the zeros of $v(N+1;\lambda)$ determine $b(n)$.

Proof. As we have seen, the polynomial $v(N;\lambda)$ has $N-1$ zeros counting multiplicities. Hence, from its zeros we also know N. Then, we also know $\sum_{j=1}^{N-1} b(j)$. Likewise, from the zeros of $v(N+1;\lambda)$ we can recover $\sum_{j=1}^{N} b(n)$. Hence, from the given data we can get $b(N)$. Having $b(N)$ we can use the difference equation, satisfied by $v(n;\lambda)$, in order to recover $v(N-1;\lambda)$. Having now $v(N;\lambda)$ and $v(N-1;\lambda)$ we can recover $b(N-1)$ and $v(N-2;\lambda)$. We continue in the same manner until we recover $b(j)$ for all $j = 1, \ldots, N$. ■

The proposition can be viewed as a special case of a discrete counterpart of a general result of Levitan and Gasymov, in the continuous case, which says that a potential can be recovered from two spectra.

(Counter)Example. Let us take $N = 4$ and consider the case $a(n) \equiv -1$ and $b(n)$ such that

$$b(1) = b(4) = \alpha + \sigma\sqrt{2}, \qquad b(2) = b(3) = \alpha - \sigma\frac{\sqrt{2}}{2}$$

where α is any fixed real (or complex) number and $\sigma \in \{-1, 1\}$. Then,

$$v(4;\lambda) = -u(5;\lambda) = -\lambda^3 + 3\alpha\lambda^2 - \left(3\alpha^2 - \frac{7}{2}\right)\lambda + \alpha^3 - \frac{7\alpha}{2}.$$

Hence, the sign σ cannot be recovered from $v(4;\lambda)$ and $u(5;\lambda)$. In other words, there are two different potentials of period $N = 4$ corresponding to the same spectral data $\{v(4;\lambda), u(5;\lambda)\}$.

We, now, wish to consider the following question: Suppose we are given a polynomial

$$\Delta(\lambda) = (-1)^N \lambda^N + \sum_{k=0}^{N-1} c_k \lambda^k.$$

Is there an N-periodic operator L_{Schr} whose discrete Hill discriminant is the given polynomial $\Delta(\lambda)$?

Let us first give a lemma of algebraic flavor.

Lemma 1. For $k = 1, \ldots, N$ let $S_k(x_1, \ldots, x_N)$ be the elementary symmetric polynomial in the variables $x_1, \ldots, x_N$ of degree k. Also, let $p_k(x_1, \ldots, x_N)$, $k = 1, \ldots, N$, be N given polynomials in $x_1, \ldots, x_N$ such that $\deg p_k \leq k-1$. Then, the cardinality of the set Λ of the distinct solutions $(x_1, \ldots, x_N) \in \mathbb{C}^N$ of the system of N equations

$$S_k(x_1, \ldots, x_N) = p_k(x_1, \ldots, x_N), \qquad k = 1, \ldots, N,$$

satisfies $1 \leq \#(\Lambda) \leq N!$.

The result follows from the simple observation that the system does not have solutions at infinity and, consequently, Λ is a compact subset of $\mathbb{C}^N$. But, then, by the Noether's Normalization Theorem we can conclude that Λ must be a finite set. Thus, the proof is finished by invoking Bézout's Theorem.

Theorem 1. Let $c_0, \ldots, c_{N-1}$ be given complex numbers. Then, there exist at least one and at most $N!$ different N-periodic potentials $b(n)$ for which the discrete Hill discriminant of the corresponding operator L_{Schr} is

$$\Delta(\lambda) = (-1)^N \lambda^N + \sum_{k=0}^{N-1} c_k \lambda^k.$$

18 Periodic Jacobi Operators Whose Spectrum is a Closed Interval

Theorem 2. Suppose that the multiplier $\rho(\lambda)$ has exactly two branch points $\eta, \theta \in \mathbb{C}$. Then, η and θ are periodic or antiperiodic eigenvalues of L satisfying

$$\left(\frac{\eta-\theta}{4}\right)^N = \pm 1$$

(in particular $|\eta - \theta| = 4$) and the spectrum of L is the line segment joining η and θ, namely

$$\sigma(L) = \{\lambda \in \mathbb{C} \,:\, \lambda = \eta + (\theta - \eta)\, t, \quad 0 \leq t \leq 1\}.$$

Theorem 3. Suppose that the spectrum $\sigma(L)$ is a simple piecewise smooth arc in the complex plane joining two (distinct) numbers η and θ. Then η and θ are the only branch points of the multiplier $r(\lambda)$. Consequently, due to Theorem 1, $\sigma(L)$ must be the line segment joining them.

Example. (i) If $N = 2$, $a(n) = i(-1)^n$, and $b(n) = 2(-1)^n$, then $\sigma(L) = [-2, 2]$.
(ii) If $N = 4$, $a(n) = (1 + i)\, i^n/\sqrt{2}$, and $b(n) = (-1)^n\sqrt{2}$, then, again, $\sigma(L) = [-2, 2]$.

18.1 Examples of Discrete Schrödinger Operators Whose Spectrum is the Interval $[-2, 2]$

Example. (i) For $N = 2$ (so that $\tilde{\Delta}_2(\lambda) = \lambda^2 - 2$) and $N = 3$ (so that $\tilde{\Delta}_3(\lambda) = -\lambda^3 + 3\lambda$) it is easy to check that $b(n) \equiv 0$.

(ii) For $N = 4$ (so that $\tilde{\Delta}_4(\lambda) = \lambda^4 - 4\lambda^2 + 2$) the system becomes

$$b_1 + b_2 + b_3 + b_4 = 0,$$

$$b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 = 0,$$

$$b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4 = 0,$$

$$b_1 b_2 b_3 b_4 = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_1.$$

It follows that b_1, b_2, b_3, and b_4 are the roots of the equation $x^4 + \alpha = 0$, where

$$\alpha = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_1.$$

Writing $b_1, b_2, b_3, b_4 = \pm(1 \pm i)\, \alpha^{1/4}/\sqrt{2}$ and substituting in the above equation yields $\alpha = 0$ or $\alpha = 4$. From the value $\alpha = 0$ we only get the obvious solution $b(n) \equiv 0$, whereas the value $\alpha = 4$ yields a total of eight distinct solutions:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1+i \\ 1-i \\ -1+i \\ -1-i \end{bmatrix}, \begin{bmatrix} -1-i \\ 1+i \\ 1-i \\ -1+i \end{bmatrix}, \begin{bmatrix} -1+i \\ -1-i \\ 1+i \\ 1-i \end{bmatrix}, \begin{bmatrix} 1-i \\ -1+i \\ -1-i \\ 1+i \end{bmatrix}$$

and

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1-i \\ 1+i \\ -1-i \\ -1+i \end{bmatrix}, \begin{bmatrix} -1+i \\ 1-i \\ 1+i \\ -1-i \end{bmatrix}, \begin{bmatrix} -1-i \\ -1+i \\ 1-i \\ 1+i \end{bmatrix}, \begin{bmatrix} 1+i \\ -1-i \\ -1+i \\ 1-i \end{bmatrix}.$$

Notice that the last three solutions are the cyclic permutations of the first solution, while the last four solutions are the complex conjugates of the first solutions. The first solution corresponds to the potential

$$b(n) = -\frac{1+i}{2}\, i^n - i(-1)^n - \frac{1-i}{2}\, (-i)^n,$$

while the other seven solutions correspond to the shifts of this potential, namely $b^1(n)$, $b^2(n)$, and $b^3(n)$, and to the complex conjugates of those four potentials (changing $b(n)$ to $b^\sharp(n) = b(-n)$ does not produce any new solutions). All these eight potentials, as well as the trivial potential $b(n) \equiv 0$ have spectrum $[-2, 2]$. Thus, there are only nine distinct solutions, while $4! = 24$.

(iii) For $N = 5$ (so that $\tilde{\Delta}_5(\lambda) = -\lambda^5 + 5\lambda^3 - 5\lambda$) the system becomes

$$S_1(b_1, b_2, b_3, b_4, b_5) = 0,$$

$$S_2(b_1, b_2, b_3, b_4, b_5) = 0,$$
$$S_3(b_1, b_2, b_3, b_4, b_5) = 0,$$
$$S_4(b_1, b_2, b_3, b_4, b_5) = b_1b_2 + b_2b_3 + b_3b_4 + b_4b_5 + b_5b_1,$$
$$S_5(b_1, b_2, b_3, b_4, b_5) = b_1b_2b_3 + b_2b_3b_4 + b_3b_4b_5 + b_4b_5b_1 + b_5b_1b_2.$$

We can find some (nontrivial) solutions by looking for solutions such that $b_j = 0$ for some j, say $b_5 = 0$. Then, the system becomes

$$S_1(b_1, b_2, b_3, b_4) = S_2(b_1, b_2, b_3, b_4) = S_3(b_1, b_2, b_3, b_4) = 0,$$

$$b_1b_2b_3b_4 = b_1b_2 + b_2b_3 + b_3b_4,$$
$$b_1b_2b_3 + b_2b_3b_4 = 0.$$

(although we have five equations with four unknowns, as we will see the resulting system has nine distinct solutions). If $b_2b_3 = 0$, then we must have $b_j = 0$ for all $j = 1, \ldots, 5$. Thus, let us assume $b_2b_3 \neq 0$. In this case the last equation of the system can be simplified as

$$b_1 + b_4 = 0.$$

As in the case (ii) it follows that b_1, b_2, b_3, and b_4 are the roots of the equation $x^4 + \alpha = 0$, where

$$\alpha = b_1b_2 + b_2b_3 + b_3b_4.$$

Writing $b_1, b_2, b_3, b_4 = \pm(1 \pm i)\, \alpha^{1/4}/\sqrt{2}$ and substituting above yields $\alpha = 0$, $\alpha = 3 + 4i$, or $\alpha = 3 - 4i$. From the value $\alpha = 0$ we only get the obvious solution $b(n) \equiv 0$. The value $\alpha = 3 + 4i$ yields the solutions:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} (1+i)\rho \\ (1-i)\rho \\ -(1-i)\rho \\ -(1+i)\rho \\ 0 \end{bmatrix}, \begin{bmatrix} (1-i)\rho \\ -(1+i)\rho \\ (1+i)\rho \\ -(1-i)\rho \\ 0 \end{bmatrix}, \begin{bmatrix} -(1-i)\rho \\ (1+i)\rho \\ -(1+i)\rho \\ (1-i)\rho \\ 0 \end{bmatrix}, \begin{bmatrix} -(1+i)\rho \\ -(1-i)\rho \\ (1-i)\rho \\ (1+i)\rho \\ 0 \end{bmatrix},$$

where

$$\rho := \frac{\sqrt{\sqrt{5}+2}}{2} + \frac{\sqrt{\sqrt{5}-2}}{2}\, i.$$

From the value $\alpha = 3 - 4i$ we get another set of four solutions, which are the complex conjugates of the above solutions.

An amusing observation is that these eight solutions can be also expressed as

$$b_1 = \pm\frac{1}{\sqrt{\phi}} \pm i\sqrt{\phi}, \qquad b_2 = \pm i b_1, \qquad b_3 = -b_2, \qquad b_4 = -b_1, \qquad b_5 = 0$$

(for all eight different choices of the plus/minus signs), where ϕ is the golden ratio, i.e.

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Finally, the cyclic permutations of the solutions and their complex conjugates produce a set of thirty two new solutions. The transformation $b(n) \to b(-n)$ does not yield any new solutions. Thus we have found a total of forty distinct solutions, plus the obvious (trivial) solution $b(n) \equiv 0$. Here we do not claim that we have found all the distinct solutions (since $5! = 120$, it is possible that more solutions exist).

18.2 The General Operator

We now consider again the more general Jacobi operator L. We are interested in the case where the spectrum $\sigma(L)$ is a simple piecewise smooth arc in the complex plane joining the numbers η and θ. Then, by Theorem 2 we can assume, essentially without loss of generality that $\sigma(L) = [-2, 2]$.
The following theorem is the discrete analog of a result of V. Guillemin and A. Uribe.

Theorem 4. Suppose that $\sigma(L) = [-2, 2]$. Then, the eigenvalues of $L|_{\mathcal{P}_{2N}}$, where $\mathcal{P}_{2N}$ is the vector space of $2N$-periodic sequences, or, equivalently, the eigenvalues of the matrix L_{2N}, are

$$\lambda_k = \tilde{\lambda}_k = -2\cos\left(\frac{\pi k}{N}\right), \qquad k = 0, 1, \ldots, N.$$

Furthermore, for $k = 1, \ldots, N-1$, either there are two linearly independent eigenfunctions (in $\mathcal{P}_{2N}$) corresponding to the eigenvalue λ_k or there is a two-dimensional generalized eigenspace (subspace of $\mathcal{P}_{2N}$) of $L|_{\mathcal{P}_{2N}}$ associated to λ_k.

Remark. A side product of Theorem 4 is that if $\sigma(L) = [-2, 2]$, then for $\lambda = -2$ and $\lambda = 2$ there is only one Floquet solution and consequently the Floquet matrices $S(-2)$ and $S(2)$ have a Jordan anomaly (at the same time, $r(\lambda)$ has a branch point at $\lambda = \pm 2$; recall, also, that $r(-2) = 1$ and $r(2) = (-1)^N$).

$\diamondsuit$

Example 4. Regarding the case $N = 4$: In the unperturbed case the matrix L_8 is similar to the diagonal matrix diag $[-2, -\sqrt{2}, -\sqrt{2}, 0, 0, \sqrt{2}, \sqrt{2}, 2]$. As for the eight cases presented in our Example for $N = 4$, the associated matrix L_8 is similar to the Jordan canonical matrix

$$\begin{bmatrix} \mathbf{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\mathbf{2}} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & -\sqrt{\mathbf{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\mathbf{2}} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \sqrt{\mathbf{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} \end{bmatrix}.$$

Finally, we present a Borg-type theorem for the general operator L with complex coefficients.

Theorem 5. Suppose that $\sigma(L) = [-2, 2]$ and that the matrix L_{2N} is diagonalizable (i.e. it has $2N$ linearly independent pure eigenvectors). Then:
(i) If N is odd, we must have $b(n) \equiv 0$ and $a(n)^2 \equiv 1$, i.e. L is an essentially unperturbed operator.
(ii) If N is even, say $N = 2M$, then $b(n) \equiv 0$ and $a(n)^2 = 1 + (-1)^n s$, where $s^2 = 1 - e^{2k\pi i/M}$ for some $k \in \{0, 1, \ldots, M-1\}$.

Notice that in the case where $N = 2M$ the theorem implies that $a(n+2)^2 = a(n)^2$ for all $n \in \mathbb{Z}$.
Theorem 5 has a nice corollary (which is essentially not new).

Corollary. If $a(n)$ and $b(n)$ are real-valued (equivalently, if L is self-adjoint) and $\sigma(L) = [-2, 2]$, then $b(n) \equiv 0$ and $a(n)^2 \equiv 1$, i.e. L is an essentially unperturbed operator.

Proof. For real-valued $a(n)$ and $b(n)$ the matrix L_{2N} of (**??**) is real symmetric and hence diagonalizable. Therefore, the corollary follows immediately from Theorem 5 since, even in the case $N = 2M$, the assumption that $a(n)$ is real forces s to be 0 (if s^2 is real, then $s^2 = 0$ or $s^2 = 2$; however, the latter cannot happen since it would make $a(n)^2$ strictly negative for certain values of n). ■

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