On inverse problems for operators with frozen argument

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Sturm–Liouville operator with frozen argument

Consider the operator

$$\mathcal{L}y := -y''(x) + p(x)y(x) + y(a)q(x), \quad x \in (0,\pi), \tag{1}$$

$$y \in W_2^2[0,\pi], \quad y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0, \quad \alpha, \beta \in \{0,1\}. \tag{2}$$

The coefficients $p, q \in L_2(0,\pi)$ are complex-valued. The value $a \in [0,\pi]$ is fixed.

Operators $\mathcal{L}$ are called Sturm–Liouville operators with *frozen argument*. Unlike classical differential operators, they are *non-local*. By this reason, study of operators with frozen argument requires development of non-classical methods.

Comment

Non-locality is the opposite property to locality. Operator is local if the value of the output function $\mathcal{L}y$ at $x \in (0,\pi)$ depends only on the values of the input function in arbitrarily small vicinity $(x-\varepsilon, x+\varepsilon)$.

Non-local operators

Non-local operators appear in models of feedback-like phenomena. Consider an example from [Kr].

For a vibrating wire of finite length l, denote by $u(x,t)$ the lateral displacement at x at time t. Let the wire be affected by a magnetic field acting with a force $K(x)[cu(0,t)+du(l,t)]$ per unit mass. The wave equation takes the form

$$u_{tt} = s^2 u_{xx} + K(x)[cu(0,t)+du(l,t)]. \tag{3}$$

Separation of variables yields the equation

$$-y''(x) - \frac{K(x)}{s^2}(cy(0)+dy(l)) = \frac{\lambda}{s^2}y(x). \tag{4}$$

Kr Krall A.M. The development of general differential and general differential-boundary systems, *Rocky Mountain J. Math.* (1975), 493–542.

Operators with frozen arguments are closely related to operators with integral boundary conditions.

Let frozen argument belong to one of the ends of the interval. For definiteness, $a = 0$, $\alpha = 1$, $\beta = 0$. Then, the adjoint operator to $\mathcal{L}y$ in $L_2(0, \pi)$ is given by

$$\mathcal{L}^* y = -\varphi''(x) + \overline{p}(x)\varphi(x), \quad x \in (0, \pi), \tag{5}$$

$$\varphi \in W_2^2(0, \pi], \quad \varphi(\pi) = 0, \tag{6}$$

$$\varphi'(0+0) = \int_0^\pi \overline{q}(t)\varphi(t)\, dt. \tag{7}$$

Integral boundary conditions arise in studying diffusion processes, see

F. Feller W. The parabolic differential equations and the associated semi-groups of transformations. *Annals of Mathematics* (1952), pp. 468–519.

F2. Feller W. Diffusion processes in one dimension. *Translations of the AMS* (1954), pp. 1–31.

Perturbation theory point view

Consider an unperturbed operator:

$$
\begin{aligned}
&Ty = -y''(x) + p(x)y(x), \quad x \in (0,\pi), \\
&y \in W_2^2[0,\pi], \quad y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0.
\end{aligned} \tag{8}
$$

The operator T is a classical Sturm–Liouville operator with the complex-valued potential $p \in L_2(0,\pi)$.

The operator $\mathcal{L}$ is a one-dimensional perturbation of $T : \mathcal{L} = T + A$, where

$$
Ay = y(a)q(x), \quad x \in (0,\pi). \tag{9}
$$

The range of values of A is a one-dimensional subspace in $L_2(0,\pi)$.

Using results from the classical book [Kat], we obtain that the spectrum L is discrete.

Kat Kato T., Perturbation Theory for Linear Operators (1966; 1995 Springer).

Theorem 5.35

The essential spectrum is conserved under a relatively compact perturbation. More precisely, let T be closed and let A be T-compact. Then T and $T + A$ have the same essential spectrum.

The Sturm–Liouville operator T and $Ay = y(a)q(x)$ satisfies the conditions of the theorem, and the essential spectrum of T is empty. (In fact, A is T-bounded and finite-dimensional $\Rightarrow$ A is T-compact).

Problem statement

Further, we study the eigenvalues $\{\lambda_n\}_{n \ge 1}$ of the BVP

$$\begin{gathered} -y''(x) + p(x)y(x) + q(x)y(a) = \lambda y(x), \quad x \in (0, \pi), \\ y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0. \end{gathered} \tag{10}$$

We aim to study an inverse problem.

Inverse problem 1.

The numbers $\alpha, \beta \in \{0, 1\}$ and $p \in L_2(0, \pi)$ are known.
Given the spectrum $\{\lambda_n\}_{n \ge 1}$, recover $q \in L_2(0, \pi)$.

In other words, we recover the perturbation A by the spectrum of $T + A$ if the unperturbed operator T is known. Note that the eigenvalues can be multiple. We assume that each λ_k appears in the spectrum $\{\lambda_n\}_{n \ge 1}$ as many times as its multiplicity.

History note

First, inverse spectral problems were studied for $p = 0$, when

$$\mathcal{L}y = -y''(x) + y(a)q(x), \quad x \in (0, l). \tag{11}$$

The problem of recovering the operator by its spectrum was addressed by many authors. A uniqueness of recovering depends on the value a and on the boundary conditions.

Rational case

Inverse problem 1 in the particular case $p = 0$ was studied completely. First, results were obtained in the rational case $a/l \in \mathbb{Q}$:

BBV Bondarenko N.P., Buterin S.A., Vasiliev S.V. An inverse spectral problem for Sturm–Liouville operators with frozen argument, J. Math. Anal. Appl. (2019).

BV Buterin S.A., Vasiliev S.V. On recovering a Sturm-Liouville-type operator with the frozen argument rationally proportioned to the interval length, J. Inv. Ill-posed Probl. (2019).

BK Buterin S., Kuznetsova M. On the inverse problem for Sturm–Liouville-type operators with frozen argument: rational case, Comp. Appl. Math. (2020).

In [BBV,BV,BK], all rational cases were studied, from the less general assumptions to the most general ones.

The approach to the inverse problem involves construction of the entire characteristic function $\Delta(\lambda)$ by its zeros $\{\lambda_n\}_{n\geq 1}$. It allows to recover some function $W \in L_2(0, l)$ that contains information on $q(x)$ as its linear transform. The potential q is recovered uniquely if and only if the linear transform is non-degenerate, which depends on a and the boundary conditions. The complete characterization of degenerate and non-degenerate combinations (a, α, β) was given in [BK]. An alternative approach to the linear transform was developed in [Ts].

BK Buterin S., Kuznetsova M. On the inverse problem for Sturm–Liouville-type operators with frozen argument: rational case, Comp. Appl. Math. (2020).

Ts Tsai T-M, Liu H-F, Buterin S, Chen L-H, Shieh C-T. Sturm–Liouville-type operators with frozen argument and Chebyshev polynomials. Math Meth Appl Sci. (2022)

Classification of rational cases

Let

$$\frac{a}{l} =: \frac{j}{k}, \quad j,k \in \mathbb{N}, \quad \gcd(j,k) = 1.$$

Degenerate cases (non-uniqueness):

$$\left.\begin{aligned} &\alpha = \beta = 0; \\ &\alpha = 1,\ \beta = 0 \text{ and } k + j \text{ even}; \\ &\alpha = \beta = 1 \text{ and } k \text{ even}; \\ &\alpha = 0,\ \beta = 1 \text{ and } j \text{ even}. \end{aligned}\right\} \tag{12}$$

Non-degenerate cases (uniqueness):

$$\left.\begin{aligned} &\alpha = 0,\ \beta = 1 \text{ and } j \text{ odd}; \\ &\alpha = 1,\ \beta = 0 \text{ and } k + j \text{ odd}; \\ &\alpha = \beta = 1 \text{ and } k \text{ odd}. \end{aligned}\right\} \tag{13}$$

Irrational case

The approach from [BBV,BV,BK,Ts] is inapplicable to the irrational a/l. Wang, Zhang, Zhao, and Wei [W] have proved that in this case, there is always uniqueness. However, the question of the necessary and sufficient conditions was left open.

W Wang Y.P, Zhang M., Zhao W., Wei X. Reconstruction for Sturm–Liouville operators with frozen argument for irrational cases, Applied Mathematics Letters (2021).

A new unified approach to the both cases was developed in [K1]. The main idea is to substitute into the characteristic function $\Delta(\lambda)$ the eigenvalues of the unperturbed operator $\lambda = \mu_n$. It allows obtaining necessary and sufficient conditions along with the stability results, see [K2].

K1 Kuznetsova M. Necessary and sufficient conditions for the spectra of the Sturm–Liouville operators with frozen argument, Applied Mathematics Letters (2022).

K2 Kuznetsova M. Uniform stability of recovering the Sturm-Liouville operators with frozen argument, Results in Mathematics (2023).

The mentioned approach can be generalized for study of Inverse problem 1 if $p \neq 0$. The difference is that we have to take into account the multiplicities of μ_n.

Another unified approach was offered in the work of Dobosevych and Hryniv:

DH Dobosevych O., Hryniv R. Reconstruction of differential operators with frozen argument, Axioms (2022).

They considered the operator with frozen argument within the framework of the perturbation theory. Their approach essentially relies on the self-adjointness of the unperturbed operator T. It is inapplicable for studying our situation, since the function p is complex-valued.

Now, we proceed to studying the inverse problem by the spectrum of the BVP $\mathcal{L}$:

$$-y''(x) + p(x)y(x) + q(x)y(a) = \lambda y(x), \quad x \in (0, \pi), \tag{14}$$

$$y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0, \tag{15}$$

in the general case $p \neq 0$. We will rely on some ideas from [K1–K2]. For simplicity, we put $\alpha = \beta = 0$ (the other boundary conditions are considered analogously).

Preliminaries

First, we introduce auxiliary objects related to the unperturbed BVP $\mathcal{L}_0$:

$$-y''(x) + p(x)y(x) = \lambda y(x), \quad x \in (0, \pi), \tag{16}$$

$$y(0) = y(\pi) = 0.$$

Denote by $S_a(x, \lambda)$ and $C_a(x, \lambda)$ the solutions of equation (16) under the initial conditions

$$C_a(a, \lambda) = 1,\ C_a'(a, \lambda) = 0, \quad S_a(a, \lambda) = 0,\ S_a'(a, \lambda) = 1. \tag{17}$$

The characteristic function of $\mathcal{L}_0$ is

$$\Delta_0(\lambda) = C_a(0, \lambda)S_a(\pi, \lambda) - S_a(0, \lambda)C_a(\pi, \lambda). \tag{18}$$

A number μ_n is an eigenvalue of $\mathcal{L}_0$ if and only if $\Delta_0(\mu_n) = 0$. Denote by $\{\mu_n\}_{n\geq 1}$ its spectrum, i.e. the sequence of the eigenvalues taken with the account of algebraic multiplicities.

The following asymptotics is known:

$$\mu_n = \theta_n^2, \quad \theta_n = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n},\ n \geq 1, \quad \{\kappa_n\}_{n\geq 1} \in \ell_2. \tag{19}$$

By m_n we denote multiplicity of the eigenvalue μ_n. By asymptotics (19), for a sufficiently large n, we have $m_n = 1$. Without loss of generality, we assume that equal values in the spectrum follow each other:

$$\begin{gathered} \mu_n = \mu_{n+1} = \ldots = \mu_{n+m_n-1},\ n \in \mathcal{S}, \\ \mathcal{S} := \{n \geq 2 \colon \mu_n \neq \mu_{n-1}\} \cup \{1\}. \end{gathered} \tag{20}$$

The index $n \in \mathcal{S}$ corresponds to the unique elements in $\{\mu_n\}_{n\geq 1}$. The index $k = n + \nu$ runs through $\mathbb{N}$, if $n \in \mathcal{S}$ and $\nu \in \overline{0, m_n - 1}$.

Example

Let $\mu_1 = \mu_2 = \mu_3 = i$ and $\mu_n = n^2$ for $n \geq 4$. Then, we have

$$\mathcal{S} = \{1, 4, 5, 6, \ldots\}, \quad m_1 = 3,\ m_4 = m_5 = m_6 = \ldots = 1.$$

For any valid example, $\mathcal{S}$ is obtained from $\mathbb{N}$ by exclusion of a finite number of elements.

Characteristic function of $\mathcal{L}$

Characteristic function of the BVP with frozen argument has the form

$$\Delta(\lambda) = \Delta_0(\lambda) - S_a(\pi, \lambda) \int_0^a W(0, t, \lambda)\, q(t)\, dt - S_a(0, \lambda) \int_a^\pi W(\pi, t, \lambda)\, q(t)\, dt, \quad (21)$$

where $W(x, t, \lambda) := C_a(t, \lambda) S_a(x, \lambda) - C_a(x, \lambda) S_a(t, \lambda)$. The spectrum $\{\lambda_n\}_{n \geq 1}$ of the BVP $\mathcal{L}$ is a sequence of the zeroes of $\Delta(\lambda)$ taken with the account of multiplicities.

Using the known asymptotics for $S_a(x, \lambda)$ and $C_a(x, \lambda)$, by Rouche's theorem, we obtain that

$$\lambda_n = \mu_n + o(n), \quad n \geq 1. \quad (22)$$

Necessary and sufficient conditions on the spectrum

The previous formula is only an approximation to the necessary and sufficient conditions. We should obtain the more precise formula

$$\lambda_n = \mu_n + b_n \varkappa_n, \; n \geq 1, \quad \{\varkappa_n\}_{n \geq 1} \in \ell_2, \tag{23}$$

where $\{b_n\}_{n \geq 1}$ is a certain sequence depending on the BVP $\mathcal{L}_0$. This sequence is bounded, but its members can be arbitrarily close to 0. In particular, if $b_n = 0$, then $\lambda_n = \mu_n$, and this eigenvalue does not depend on q. We say that λ_n degenerates, because it brings no information on q. It can also occur that

$$b_n = b_{n+1} = \ldots = b_{n+k_n-1} = 0,$$

$$\mu_n = \mu_{n+1} = \ldots = \mu_{n+k_n-1}, \quad k_n > 1.$$

In this case, formula (23) gives degeneration condition for k_n subsequent eigenvalues $\lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+k_n-1} = \mu_n$. We will have that μ_n is an eigenvalue of $\mathcal{L}$ with the multiplicity not less than k_n.

First, we introduce a sequence $\{a_n\}_{n\geq 1}$:

$$a_{n+\nu} = \frac{n}{\nu!} \left. \frac{\partial^\nu S_a(\pi,\lambda)}{\partial \lambda^\nu} \right|_{\lambda=\mu_n}, \quad n \in \mathcal{S},\ \nu = \overline{0, m_n - 1}. \tag{24}$$

Remind that $S_a(x,\lambda)$ is the solution of the initial-value problem

$$-y'' + p(x)y = \lambda y, \quad y(a) = 0,\ y'(a) = 1 \tag{25}$$

(in other words, $S_a(\pi,\lambda)$ is the characteristic function for the unperturbed BVP on (a,π)).

Let r_n be a multiplicity of μ_n as a zero of $S_a(\pi,\lambda)$. Put

$$b_{n+\nu} = \begin{cases} a_{n+\nu}, & \nu = \overline{0, p_n - 1}, \\ 1, & \nu = \overline{p_n, m_n - 1}, \end{cases} \quad p_n := \max\left(1, \min(r_n, m_n)\right), \quad n \in \mathcal{S}. \tag{26}$$

In fact, $b_{n+\nu}$ differs from $a_{n+\nu}$ only in finite number of elements corresponding to $m_n > 1$.

Example

Let $\mu_1 = \mu_2 = \mu_3 = i$ and $\mu_n = n^2$ for $n \geq 4$. Then, we have

$$\mathcal{S} = \{1, 4, 5, 6, \ldots\}, \quad m_1 = 3,\ m_4 = m_5 = m_6 = \ldots = 1.$$

Thus, for $K = 4$ and $n \geq K$, all $m_n = 1$.

$$a_1 = S_a(\pi, i),\ a_2 = \frac{\partial S_a(\pi, i)}{\partial \lambda},\ a_3 = \frac{\partial^2 S_a(\pi, i)}{\partial \lambda^2};$$

$$a_n = nS_a(\pi, n^2),\ n \geq 4.$$

Let $\lambda = i$ be a zero of $S_a(\pi, \lambda)$ of order $r_1 = 1$.
We have $a_1 = 0$, $a_2 \neq 0$, while a_3 can be arbitrary. Then,

$$p_1 = 1, \quad p_n = 1,\ n \geq 4,$$

$$b_1 = a_1 = 0,\ b_2 = b_3 = 1; \quad b_n = a_n,\ n \geq 4.$$

Theorem 1 (necessary and sufficient conditions)

For an arbitrary sequence $\{\lambda_n\}_{n\ge 1}$ of complex numbers to be the spectrum of the BVP $\mathcal{L}$ with some $q \in L_2(0,\pi)$, it is necessary and sufficient to satisfy the formula

$$\lambda_n = \mu_n + b_n \varkappa_n, \; n \ge 1, \quad \{\varkappa_n\}_{n\ge 1} \in \ell_2. \tag{27}$$

Thus, for $n \in \Omega$, the eigenvalue λ_n degenerates,

$$\Omega = \{n \in \mathbb{N} \colon b_n = 0\} = \{n + \nu \colon n \in \mathcal{S}, \; 0 \le \nu < \min(m_n, r_n)\}.$$

The necessity part is proved by accurate application of asymptotic expansions. The proof of the sufficiency part is constructive. We consider the procedure of recovering q from the spectrum satisfying the condition (27). This procedure allows us to investigate the uniqueness and to find which additional data are needed in the case of non-uniqueness.

Procedure of recovering

For $n \in \mathcal{S}$ and $\nu = \overline{0, m_n - 1}$, we introduce

$$g_{n+\nu}(t) = \frac{n^{1-\alpha}}{(m_n - \nu - 1)!} \left. \frac{\partial^{m_n-\nu-1} g(t,\lambda)}{\partial \lambda^{m_n-\nu-1}} \right|_{\lambda=\mu_n}, \qquad (28)$$

where $g(x,\lambda)$ is a solution of the initial-value problem

$$-y''(x) + p(x)y(x) = \lambda y(x),\ x \in (0,\pi), \quad y(0) = 0,\ y'(0) = 1.$$

It is known that

$$g(x,\rho^2) = \frac{\sin \rho x}{\rho} + O\left(\frac{e^{|\mathrm{Im}\,\rho x|}}{\rho}\right).$$

Then, the sequence $\{g_n(t)\}_{n\geq 1}$ is an almost normalized sequence of eigen- and associated functions of the operator $Ty = -y'' + p(x)y$ with Dirichlet boundary conditions.

Proposition

The functional sequence $\{g_n(t)\}_{n\geq 1}$ is a Riesz basis in $L_2(0,\pi)$.

By the spectrum, we can uniquely reconstruct the characteristic function:

$$\Delta(\lambda) = \pi \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{k^2}. \tag{29}$$

On the other side,

$$\Delta(\lambda) = \Delta_0(\lambda) - S_a(\pi, \lambda) \int_0^a W(0, t, \lambda)\, q(t)\, dt - S_a(0, \lambda) \int_a^{\pi} W(\pi, t, \lambda)\, q(t)\, dt. \tag{30}$$

For each $n \in \mathcal{S}$, we differentiate the both parts of the formula $\nu = \overline{0, m_n - 1}$ times and put $\lambda = \mu_n$:

$$d_{n+\nu} = \sum_{\eta=0}^{\nu} a_{n+\nu-\eta} \xi_{n+m_n-1-\eta}, \quad n \in \mathcal{S},\ \nu = \overline{0, m_n - 1}, \tag{31}$$

where

$$d_{n+\nu} = n^2 \frac{\Delta^{(\nu)}(\mu_n)}{\nu!}, \quad \xi_k := \int_0^{\pi} g_k(t) q(t)\, dt.$$

Recovering the coefficients $\xi_n := \int_0^\pi g_n(t)q(t)\,dt$, $n \in \mathbb{N}$, is equivalent to recovering the potential q. Since $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz basis, for any sequence $\{\xi_n\}_{n\in\mathbb{N}} \in \ell_2$, there corresponds a valid $q \in L_2(0,\pi)$.

One can see that in the relation

$$d_{n+\nu} = \sum_{\eta=0}^{\nu} a_{n+\nu-\eta}\xi_{n+m_n-1-\eta}, \quad n \in \mathcal{S},\ \nu = \overline{0, m_n - 1}, \tag{32}$$

the values a_k and d_k are known (a_k are constructed by the unperturbed BVP, while d_k are constructed by the characteristic function).

For each fixed $n \in \mathcal{S}$, we can consider (32) as a system of m_n linear algebraic equations with respect to $\xi_n, \xi_{n+1}, \dots, \xi_{n+m_n-1}$. Its unique solvability depends on the properties of the coefficients $a_n, a_{n+1}, \dots, a_{n+m_n-1}$.

For each fixed $n \in \mathcal{S}$, (32) has the form

$$\begin{bmatrix} 0 & \dots & 0 & 0 & 0 & a_n \\ 0 & \dots & 0 & 0 & a_n & a_{n+1} \\ 0 & \dots & 0 & a_n & a_{n+1} & a_{n+2} \\ \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\ a_n & \dots & \dots & a_{m_n-3} & a_{m_n-2} & a_{m_n-1} \end{bmatrix} \begin{bmatrix} \xi_n \\ \xi_{n+1} \\ \xi_{n+2} \\ \vdots \\ \xi_{n+m_n-1} \end{bmatrix} = \begin{bmatrix} d_n \\ d_{n+1} \\ d_{n+2} \\ \vdots \\ d_{n+m_n-1} \end{bmatrix}$$

with a triangular matrix, where auxiliary diagonals contain equal numbers.
If μ_n is a zero of $S_a(\pi, \lambda)$ of multiplicity $r_n \geq m_n$, then

$$a_n = a_{n+1} = \dots = a_{n+m_n-1} = 0.$$

By the necessary conditions, $d_n = d_{n+1} = \dots = d_{n+m_n-1} = 0$, and the system fully degenerates (but keeps compatibility). In this case, we have no information on $\xi_n, \xi_{n+1}, \dots, \xi_{n+m_n-1}$.

Let μ_n be a zero of $S_a(\pi, \lambda)$ of multiplicity $r_n < m_n$, then

$$a_n = a_{n+1} = \ldots = a_{n+r_n-1} = 0, \quad a_{n+r_n} \neq 0.$$

The first r_n equations in the system degenerate, and there reminds a system

$$\begin{bmatrix} 0 & \ldots & 0 & 0 & a_{n+r_n} \\ 0 & \ldots & & a_{n+r_n} & a_{n+r_n+1} \\ 0 & \ldots & a_{n+r_n} & a_{n+r_n+1} & a_{n+r_n+2} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ a_{n+r_n} & \ldots & a_{m_n-3} & a_{m_n-2} & a_{m_n-1} \end{bmatrix} \begin{bmatrix} \xi_{n+r_n} \\ \xi_{n+r_n+1} \\ \xi_{n+r_n+2} \\ \vdots \\ \xi_{n+m_n-1} \end{bmatrix} = \begin{bmatrix} d_{r_n} \\ d_{r_n+1} \\ d_{r_n+2} \\ \vdots \\ d_{n+m_n-1} \end{bmatrix}$$

which includes $m_n - r_n$ equations with respect to $\xi_{n+r_n}, \ldots, \xi_{n+m_n-1}$. These variables are uniquely determined:

$$\xi_{n+m_n-\nu} = \frac{1}{a_{n+r_n}} \left(d_{n+r_n+\nu-1} - \sum_{\eta=1}^{\nu-1} a_{n+r_n+\eta} \xi_{n+m_n-\nu+\eta} \right), \ \nu = \overline{1, m_n - r_n}.$$

The other variables $\xi_n, \ldots, \xi_{n+r_n-1}$ disappear, so they can not be found.

We write the set of indices n such that ξ_n are can not be determined:

$$\{n+\nu\colon n \in \mathcal{S},\ 0 \le \nu < \min(m_n, r_n)\} = \Omega.$$

This set has been already used to denote the degenerating part of the spectrum $\{\lambda_n\}_{n\in\Omega}$.

It turns out that a variation of $\{\xi_n\}_{n\in\Omega} \in \ell_2$ does not influence the spectrum $\{\lambda_n\}_{n\geq 1}$. Thus, for a fixed spectrum $\{\lambda_n\}_{n\geq 1}$, one can construct the set of all iso-spectral potentials q varying $\{\xi_n\}_{n\in\Omega} \in \ell_2$ or find a unique q setting additionally $\{\xi_n\}_{n\in\Omega} \in \ell_2$.

Inverse problem 2

Given the spectrum $\{\lambda_n\}_{n\notin\Omega}$ and $\{\xi_n\}_{n\in\Omega}$, recover $q \in L_2(0,\pi)$.

Consider another boundary value problem $\tilde{\mathcal{L}}$ which differs from $\mathcal{L}$ only in the potential $\tilde{q}$:

$$-y''(x) + p(x)y(x) + \tilde{q}(x)y(a) = \lambda y(x), \quad x \in (0,\pi),$$

$$y(0) = y(\pi) = 0.$$

Theorem 2 (uniqueness theorem)

Let $\{\tilde{\lambda}_n\}_{n\geq 1}$ be the spectrum of another boundary value problem $\tilde{\mathcal{L}}$ with a potential $\tilde{q} \in L_2(0,\pi)$, while $\tilde{\xi}_n = \int_0^\pi \tilde{q}(t) g_n(t)\, dt$, $n \in \Omega$. If $\{\lambda_n\}_{n\notin\Omega} = \{\tilde{\lambda}_n\}_{n\notin\Omega}$ and $\xi_n = \tilde{\xi}_n$ for $n \in \Omega$, then $q = \tilde{q}$.

Comment

The case $\Omega = \varnothing$ is admissible. Under this condition, no additional data are needed, and we have the uniqueness theorem for Inverse problem 1.

Theoretical procedure

1. Construct $\Delta(\lambda)$ via the formula $\Delta(\lambda) = \pi \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{k^2}$.
2. For $n \in \mathcal{S}$ and $\nu = \overline{0, m_n - 1}$, find the numbers

$$a_{n+\nu} = \frac{n}{\nu!} \left. \frac{\partial^\nu S_a(\pi, \lambda)}{\partial \lambda^\nu} \right|_{\lambda = \mu_n}, \quad d_{n+\nu} = n^2 \frac{\Delta^{(\nu)}(\mu_n)}{\nu!}.$$

3. For $n \in \mathcal{S}$ and $\nu = \overline{1, m_n - r_n}$, compute

$$\xi_{n+m_n-\nu} = \frac{1}{a_{n+r_n}} \left(d_{n+r_n+\nu-1} - \sum_{\eta=1}^{\nu-1} a_{n+r_n+\eta} \xi_{n+m_n-\nu+\eta} \right).$$

4. Find $q = \sum_{n=1}^{\infty} \xi_n f_n$, where $\{f_n\}_{n \geq 1}$ is the basis biorthonormal to $\{\overline{g}_n\}_{n \geq 1}$ in $L_2(0, \pi)$.

Uniform stability

Let $\{\lambda_n\}_{n\notin\Omega}$ and $\{\xi_n\}_{n\in\Omega}$ be the data for recovering the BVP $\mathcal{L}$. We proved that necessary and sufficient conditions for them are as follows:

$$\begin{gathered} \lambda_n = \mu_n + b_n \varkappa_n, \ n \notin \Omega, \\ \big\|\{\varkappa_n\}_{n\notin\Omega}\big\| < \infty, \ \big\|\{\xi_n\}_{n\in\Omega}\big\| < \infty, \end{gathered} \tag{33}$$

where $\|\cdot\|$ is the standard norm in ℓ_2 :

$$\big\|\{x_n\}_{n\in A}\big\| = \sqrt{\sum_{n\in A} |x_n|^2}.$$

Let $\{\tilde\lambda_n\}_{n\notin\Omega}$ and $\{\tilde\xi_n\}_{n\in\Omega}$ be the input data corresponding to another BVP $\tilde{\mathcal{L}}$ which differs only in the potential $\tilde q$ (see p. 28). Then, we have

$$\begin{gathered} \tilde\lambda_n = \mu_n + b_n \tilde\varkappa_n, \ n \notin \Omega, \\ \big\|\{\tilde\varkappa_n\}_{n\notin\Omega}\big\| < \infty, \ \big\|\{\tilde\xi_n\}_{n\in\Omega}\big\| < \infty. \end{gathered} \tag{34}$$

The following theorem gives the uniform stability of Inverse problem 2.

Theorem 3.

Consider arbitrary $\delta > 0$ *and two BVP* $\mathcal{L}$ *and* $\tilde{\mathcal{L}}$. *If*

$$\left\|\{\varkappa_n\}_{n\notin\Omega}\right\| \le \delta, \quad \left\|\{\tilde{\varkappa}_n\}_{n\notin\Omega}\right\| \le \delta, \tag{35}$$

then

$$\|q - \tilde{q}\|_{L_2(0,\pi)} \le C_\delta \Lambda + C\Xi, \tag{36}$$

where

$$\Lambda := \left\|\{\varkappa_n - \tilde{\varkappa}_n\}_{n\notin\Omega}\right\|, \quad \Xi := \left\|\{\xi_n - \tilde{\xi}_n\}_{n\in\Omega}\right\|.$$

The constant $C_\delta > 0$ *depends only on* p, a, *and* δ, *while* $C > 0$ *depends only on* p.

Relation to the case $p = 0$

Now, we discuss the case $p = 0$ that was studied in the previous works. The unperturbed BVP $\mathcal{L}_0$ has the form

$$-y'' = \lambda y, \quad y(0) = y(\pi) = 0.$$

The eigenvalues of this BVP are

$$\mu_n = n^2, \quad n \geq 1 \quad (m_n = 1,\ \mathcal{S} = \mathbb{N}).$$

We also have

$$S_a(x, \rho^2) = \frac{\sin \rho(x - a)}{\rho}, \quad b_n = a_n = nS_a(\pi, n^2) = (-1)^{n+1} \sin na.$$

Thus, the necessary and sufficient conditions on the spectrum are

$$\lambda_n = n^2 + \kappa_n \sin na,\ n \in \mathbb{N}, \quad \{\kappa_n\}_{n \in \mathbb{N}} \in \ell_2; \tag{37}$$

$$\Omega = \{n \in \mathbb{N}\colon\ \sin na = 0\}.$$

1) if $a/\pi = j/k$, then $\sin na = 0 \Leftrightarrow n = km$, $m \in \mathbb{N}$, and

$$\Omega = \{km\colon m \in \mathbb{N}\}.$$

One spectrum is insufficient for uniqueness, which agrees with [BV, BBV, BK] (see p. 11). We need additional data $\{\xi_{km}\}_{m\geq 1}$, where

$$\xi_n = \int_0^\pi g_n(t)q(t)\,dt, \quad g_n(t) = ng(t, n^2) = \sin nx.$$

Thus, $\xi_{km} = \int_0^\pi \sin kmt\, q(t)\,dt$.

2) if $a/\pi \notin \mathbb{Q}$ we have $\sin na \neq 0$ for any $n \in \mathbb{N}$, and $\Omega = \varnothing$. In this case, q is recovered uniquely by the spectrum $\{\lambda_n\}_{n\geq 1}$, which agrees with the result from [W].

Other directions

Finally, we briefly mention the rest bibliography related to inverse problems for operators with frozen argument.

A. Other boundary conditions

BH Buterin S., Hu Y.T. Inverse spectral problems for Hill-type operators with frozen argument, Anal. Math. Phys. (2021).

In [BH], the inverse problem for equation

$$-y''(x) + y(a)q(x) = \lambda y(x)$$

with periodic or anti-periodic boundary conditions was studied. In this case, only spectrum is not sufficient for recovering q.

As far as we know, Nizhnik, Hryniv, and Albeverio first studied an inverse spectral problem for operators with frozen argument.

Alb Albeverio S., Hryniv R.O., Nizhnik L.P. Inverse spectral problems for non-local Sturm–Liouville operators, Inverse Problems (2007).

Niz Nizhnik L.P. Inverse eigenvalue problems for nonlocal Sturm–Liouville operators, Meth. Func. Anal. Top. (2009).

Niz2 Nizhnik L.P., Inverse nonlocal Sturm-Liouville problem, Inverse Problems (2010).

They considered frozen argument in the end or in the middle of the segment. The boundary conditions contained an integral member to get a self-adjoint operator. The approach exploited theory of perturbations and the self-adjointness. It was found that in some cases, one spectrum is sufficient for recovering q, while in the other, q is not recovered uniquely.

B. Difference equations with frozen argument

Bon Bondarenko N.P. Finite-difference approximation of the inverse Sturm-Liouville problem with frozen argument, Applied Mathematics and Computations (2022).

In [Bon], a second-order difference equation with frozen argument was studied, being a discrete approximation of continuous problem. The relationship between the eigenvalues of the continuous problem and its finite-difference approximation was established and used for development of a numerical algorithm.

C. Equations with frozen argument on time scales

K3 Kuznetsova M. Inverse problem for Sturm–Liouville operators with frozen argument on closed sets, Itogi Nauki i Tekhniki. Sovr. Matematika (2022) [in Russian; English translation in arXiv:2107.05125 [math.SP] (11.07.21)].

In [K3], results were obtained for the Sturm–Liouville equation with Δ-derivatives and frozen argument.

D. Equations with frozen argument on graphs

Bon2 Bondarenko N.P. Inverse problem for a differential operator on a star-shaped graph with nonlocal matching condition, Boletín de la Sociedad Matemática Mexicana (2022).

A Sturm-Liouville operator on a star-shaped graph with non-local matching conditions at a central vertex was considered. It is adjoint to Sturm-Liouville operator on the graph with frozen argument.

E. Inverse nodal problems and traces formulae

HBY Hu Y.-T., Bondarenko N.P., Yang Ch.-F., Traces and inverse nodal problem for Sturm–Liouville operators with frozen argument, Applied Mathematics Letters (2020).

HHY Hu Y.-T., Huang Zh.-Y., Yang Ch.-F., Traces for Sturm–Liouville Operators with Frozen Argument on Star Graphs, Results in Mathematics (2020).

F. Quadratic differential pencils with frozen argument

HS Hu Y.-T., Sat M., Trace Formulae for Second-Order Differential Pencils with a Frozen Argument, Mathematics (2023).

HS2 Hu Y.-T., Sat M., Inverse spectral problem for differential pencils with a frozen argument, J. of Inverse and Ill-Posed Problems (2024).

The spectrum of the following BVP was considered:

$$-y''(x) + \rho q_1(x) y(a) + q_0(x) y(a) = \rho^2 y, \quad x \in (0, 1),$$

$$y^{(\alpha)}(0) = y^{(\beta)}(1) = 0.$$

An attempt to recover q_0 and q_1 by the spectrum was made in [HS2]. However, computations concern only a uniqueness theorem and the rational case $a = 1/k$. Moreover, the complete proof was not given.

G. Two or more frozen arguments

ST Shieh Ch.-Ts., Tsai T.-M., Inverse spectral problem of Sturm-Liouville equation with many frozen arguments, arXiv:2407.14889v1 [math.SP]

The inverse problem of recovering q by the spectrum of the following BVP was studied:

$$-y''(x) + q(x) \sum_{s=1}^{n} y(a_s) = \lambda y(x), \quad x \in (0, \pi),$$

$$y(0) = y(\pi) = 0.$$

Under some assumptions, the authors obtained a uniqueness theorem.

K5 Kuznetsova, M. On recovering non-local perturbation of non-selfadjoint Sturm-Liouville operator. Izvestiya of Saratov University. Mathematics. Mechanics. Informatics (to appear). https://doi.org/10.48550/arXiv.2307.10075

Thank you for attention!