Maximal dissipative operators on metric graphs: real eigenvalues and their multiplicities

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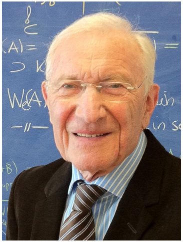
Dedicated to the memory of Heinz Langer



1. Dissipative operators

Def 1. *An operator L on a Hilbert space $\mathcal{H}$ is called* **dissipative** *if and only if it is densely defined and*

$$\mathfrak{Im}\langle u, Lu\rangle \geq 0, \qquad \forall u \in \mathrm{Dom}\,(L).$$

The semigroup

$$e^{iLt}, \quad t \geq 0$$

is contracting.

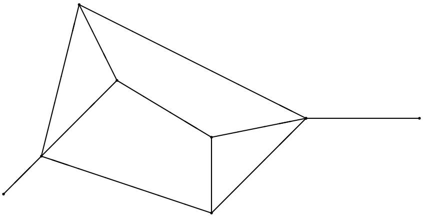
2. Metric graphs

Def 2. *A* **metric graph** *is a pair* $\Gamma = (\mathbf{E}, \mathbf{V})$ *of edges and vertices*

- $\mathbf{E} = \{e_n\}_{n=1}^N$ *finite set of intervals* $e_n = [x_{2n-1}, x_{2n}] \subset \mathbb{R}$,
- $\mathbf{V} = \{v^m\}_{m=1}^M$ *equivalence classes on the set of end points* $\mathbb{V} := \{x_j\}_{j=1}^{2N}$.

Formally

$$\Gamma := \bigsqcup_{e_n \in \mathbf{E}} \overline{e_n} \Big/ \sim .$$



3.Schrödinger operators on graphs

Schrödinger operator

$$L\psi = -\frac{d^2}{dx^2}\psi + q(x)\psi, \qquad q \in L_\infty(\Gamma),$$

where q is not assumed to be real-valued and with domain of the form

$$\mathrm{dom}(L) = \{\psi \in W_2^2(\Gamma \setminus \mathbf{V}) : A\Psi - B\partial\Psi = 0\}$$

for some $r \times d$ complex matrices A, B, where $d := 2N$ is the number of end points and $r \in \mathbb{N}$.

Ψ - the vector of limiting values of the functions at the end points
$\partial\Psi$ - the vector of limiting values of the oriented derivatives of functions at the end points

Part I. Necessary and sufficient conditions

Lemma 3. *The Schrödinger operator is* **dissipative** *if and only if both*

$$\mathfrak{Im}\, q \geq 0 \quad \text{and} \quad \mathfrak{Im}\langle \Psi, \partial \Psi \rangle_{\mathbb{C}^d} \geq 0$$

for any vector Ψ *satisfying the vertex conditions.*

$$\langle \psi, L\psi \rangle_{L_2(\Gamma)} = \int_\Gamma \left(|\psi'(x)|^2 + q(x)|\psi(x)|^2 \right) dx + \langle \Psi, \partial \Psi \rangle_{\mathbb{C}^d}$$

$$\mathfrak{Im}\langle \psi, L\psi \rangle_{L_2(\Gamma)} = \int_\Gamma \mathfrak{Im}\, q(x)|\psi(x)|^2 dx + \mathfrak{Im}\left(\langle \Psi, \partial \Psi \rangle_{\mathbb{C}^d} \right)$$

Thm 4. *The Schrödinger operator from our class is* **maximal dissipative** *if and only if*

$$\mathfrak{Im}\, q \geq 0, \quad rank\,(A|B) = N \quad \text{and} \quad \mathfrak{Im}(AB^*) \geq 0.$$

NB! Not only maximal operator among Schrödinger operators on Γ, but also maximal dissipative – no dissipative extension exists.

Important example: delta couplings with $\mathfrak{Im}\, \alpha_m \geq 0$

$$\begin{cases} x_i, x_j \in v^m \Rightarrow \psi(x_i) = \psi(x_j) & \text{– continuity condition,} \\ \displaystyle\sum_{x_i \in v^m} \partial\psi(x_i) = \alpha_m \psi(v^m) & \text{– delta condition.} \end{cases}$$

Thm 5. *The Schrödinger operator from our class is* **maximal dissipative** *if and only if*

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Part II. Completely non-self-adjoint operators

Def 6. *An operator L on a Hilbert space $\mathcal{H}$ is called* **completely non-self-adjoint** *if any only if there exists no non-trivial reducing subspace V for L such that $L|_V$ is self-adjoint.*

Proposition 1. *If every point in $\Sigma(A) \cap \mathbb{R}$ is an eigenvalue, then A possesses the following* **Langer** *decomposition:*

$$A = A|_{\mathcal{H}_{\mathrm{S}}} \oplus A|_{\mathcal{H}_{\mathrm{S}}^{\perp}} ,$$

where

$$\begin{cases} A|_{\mathcal{H}_{\mathrm{S}}} & \text{is self-adjoint,} \\ A|_{\mathcal{H}_{\mathrm{S}}^{\perp}} & \text{is completely non-self-adjoint.} \end{cases}$$

Our goal: characterise real eigenvalues of L and understand their multiplicities.

Step 1. Dissipative edges

$\psi(\lambda)$ - any eigenfunction corresponding to a real eigenvalue $\lambda \in \mathbb{R}$

$$\lambda \|u\|^2 = \langle u, Lu \rangle = \langle u, -u'' + qu \rangle = \|u'\|^2 + \int_\Gamma q(x)|u(x)|^2 dx + \langle \Psi, \partial\Psi \rangle_{\mathbb{C}^d}$$

$$\Rightarrow \int_\Gamma \mathfrak{Im}\, q(x)|u(x)|^2 dx + \underbrace{\mathfrak{Im}\langle \Psi, \partial\Psi \rangle_{\mathbb{C}^d}}_{\geq 0} = 0.$$

$$\Rightarrow \int_\Gamma \mathfrak{Im}\, q(x)|u(x)|^2 dx = 0.$$

Def 7. *An edge e_j is* **dissipative** *if there exists a subset Δ of e_j of positive Lebesgue measure $|\Delta| > 0$ on which $\mathfrak{Im}\, q \neq 0$.*

Lemma 8. *Let the edge e_j be dissipative, then let $\Gamma_j := \Gamma \setminus e_j$ be the graph formed by deleting e_j. Then*

$$\mathcal{H}_{\mathrm{S}}(L) = \mathcal{H}_{\mathrm{S}}(L|_{\Gamma_j}).$$

Dissipative edges can be ignored determining the self-adjoint part of L.
We get both Dirichlet and Kirchhoff conditions at the vertices where dissipative edges were attached.

Step 2. Dissipative vertices

$$\underbrace{\int_\Gamma \mathfrak{Im}\, q(x)|u(x)|^2 dx}_{\geq 0} + \mathfrak{Im}\langle \Psi, \partial\Psi\rangle_{\mathbb{C}^d} = 0.$$

$$\Rightarrow \int_\Gamma \mathfrak{Im}\, q(x)|u(x)|^2 dx = \sum_m \mathfrak{Im}\, \alpha_m |\psi(v^m)|^2 = 0.$$

Def 9. *A vertex v^m of Γ is* **dissipative** *if the coupling constant α_m is not real: $\mathfrak{Im}\, \alpha_m > 0$.*

Determining the self-adjoint part of L one should consider only functions equal to zero at dissipative vertices.
Such eigenfunctions satisfy both Dirichlet and Kirchhoff conditions at dissipative vertices:

$$\begin{cases} \psi(v^m) = 0 & \text{– Dirichlet condition,} \\ \sum_{x_i \in v^m} \partial\psi(x_i) = 0 & \text{– Kirchhoff condition.} \end{cases}$$

$\Rightarrow$ **Dirichlet-Kirchhoff vertices**.

The first reduction: metric graphs

- dissipative edges should be removed – every ψ should be zero there;
- Dirichlet-Kirchhoff conditions should be assumed at all dissipative vertices.

Can Dirichlet-Kirchhoff conditions appear at some other vertices?

Yes, consider two dissipative edges meeting at a degree 3 vertex, then ψ satisfies both Dirichlet and Neumann (=Kirchhoff) condition at the end point belonging to the third edge.

Def 10. *An edge e_n is* **redundant** *if and only if it has an end point which is a Dirichlet-Kirchhoff-vertex of degree* 1*.*

- All redundant edges should be removed – every ψ should be zero there;

Def 11. *By deleting dissipative and redundant edges in the original metric graph Γ, the new graph may again have redundant edges. The graph formed by repeating this procedure until it stabilises is called the* **Hermitian core of Γ (for** L**)***, and is denoted by Γ_0.*

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Hermitian core

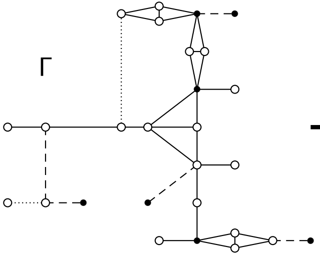


Figure: The peeling process applied to a graph Γ with dissipative edges (dotted) and dissipative vertices ($\bullet$). Redundant edges which are not dissipative are dashed. The result is the graph Γ_0.

The reduced graph can be seen as a set of smaller metric graphs glued together at Dirichlet-Kirchhoff vertices (black bullets).

Dirichlet and standard operators on the Hermitian core

Consider three Schrödinger operators determined by the same (Hermitian) differential expression on the functions satisfying the original vertex conditions at all not Dirichlet-Kirchhoff-vertices and the following conditions at Dirichlet-Kirchhoff-vertices

- $L^{\mathrm{D}}(\Gamma_0)$ – the Dirichlet conditions;
- $L^{\mathrm{st}}(\Gamma_0)$ – the standard conditions;
- $L^{\mathrm{DK}}(\Gamma_0)$ – the Dirichlet-Kirchhoff conditions.

$L^{\mathrm{D}}(\Gamma_0)$ and $L^{\mathrm{st}}(\Gamma_0)$ are self-adjoint, $L^{\mathrm{DK}}(\Gamma_0)$ - symmetric, their common restriction.

Thm 14. *L is completely non-self-adjoint if and only if there are no eigenvalues $\lambda \in \Sigma(L^{\mathrm{D}}(\Gamma_0)) \cap \Sigma(L^{\mathrm{st}}(\Gamma_0))$ such that λ-eigenspaces of $L^{\mathrm{D}}(\Gamma_0)$ and $L^{\mathrm{st}}(\Gamma_0)$ have a non-trivial intersection. The following are therefore equivalent:*

(a) *L is not completely non-self-adjoint,*

(b) *there exists a nontrivial eigenfunction of $L^{\mathrm{DK}}(\Gamma_0)$,*

(c) *$L^{\mathrm{D}}(\Gamma_0)$ and $L^{\mathrm{st}}(\Gamma_0)$ have a common eigenfunction.*

Hypergraphs

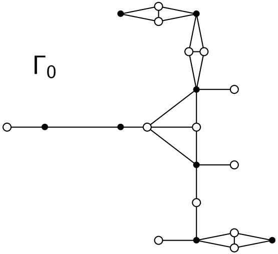


Figure: Hermitian core with Dirichlet-Kirchhoff vertices (•).

The reduced graph can be seen as a set of smaller metric graphs glued together at Dirichlet-Kirchhoff vertices.

$\Rightarrow$ The language of hypergraphs can be used.

NB! We need metric analogs of hypergraphs.

Hypergraphs (metric)

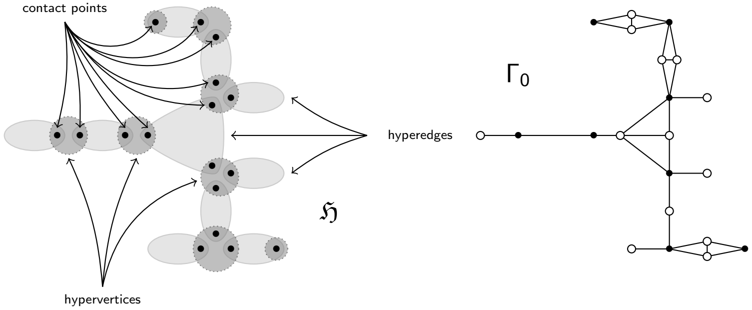


Figure: A hypergraph, with contact points ($\bullet$), hyperedges (light grey regions), and hypervertices (dark grey regions) labelled.

Def 15. *Let* $\mathbb{V}$ *be a set of* **contact points**. *A* **hypergraph** $\mathfrak{H} = (\mathfrak{E}, \mathfrak{V})$ *is a pair, where*

- $\mathfrak{E} = \{\mathfrak{e}_n\}_{n=1}^N$, *is an (hyperedge) partition of* $\mathbb{V}$ *into non-intersecting subsets called* **hyperedges**,
- $\mathfrak{V} = \{\mathfrak{v}^m\}_{m=1}^M$, *is a (hypervertex) partition of* $\mathbb{V}$ *into some equivalence classes called* **hypervertices**.

Hypercycles

Def 16. *Assume that* $\mathfrak{v}_l$, $l = 1, 2, \dots, n$ *is a sequence of hypervertices and* $\mathfrak{e}_j$, $l = 1, 2, \dots, n$ *is a sequence of hyperedges in* $\mathfrak{H}$ *such that*

- *every* $\mathfrak{v}_l$ *belongs to two consecutive hyperedges* $\mathfrak{e}_l$ *and* $\mathfrak{e}_{l+1}$;
- *every hyperedge* $\mathfrak{e}_j$ *contains* $\mathfrak{v}_{j-1}$ *and* $\mathfrak{v}_j$,

where we use cyclic notation $n + 1 = 1$. *Then the* **hypercycle** *is the hypergraph formed by* $\mathfrak{e}_j$, $l = 1, 2, \dots, n$ *with the connections determined* **only** *by* $\mathfrak{v}_l$.

NB! Hypercycle does not need to be a subhypergraph – not all connections between the hyperdeges present in the original hypergraph are preserved.

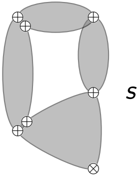


Figure: A hyper-4-cycle s.

Hypertrees

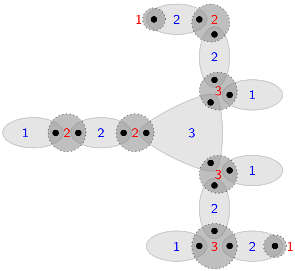
Hypertrees – hypergraphs having no hypercycles.

Proposition 2. *Let $\mathfrak{H} = (\mathfrak{E}, \mathfrak{V})$ be hyperforest. Then*

$$\sum_{deg(\mathfrak{v}^m)\geq 2} (deg(\mathfrak{v}^m) - 2) + \sum_{val(\mathfrak{e}_n)\geq 2} (val(\mathfrak{e}_n) - 2) = T - 2\beta_0(\mathfrak{H}).$$

where

- $\beta_0(\mathfrak{H})$ *– number of connected components of* $\mathfrak{H}$*,*
- T *– number of terminals,*
- $deg(\mathfrak{v}^m) = |\mathfrak{v}^m|$ *- number of points in the hypervertex;*
- $val(\mathfrak{e}_n)$ *– number of connections to the rest of the graph.*



$3 + 1 = 6 - 2 \times 1$

Branching is due to higher degree vertices and edges

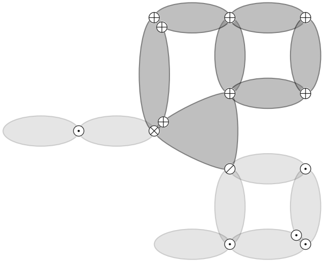
In particular we have:

$$2\beta_0(\mathfrak{H}) \leq \sum_{\mathfrak{e}_n \in \mathfrak{E}} (2 - \mathrm{val}_{\mathfrak{H}}(\mathfrak{e}_n)) \leq T.$$

- The lower bound is attained if and only if every hypervertex has hyperdegree at most 2.
- The upper bound is attained if and only if every hyperedge has valency at most 2.

Hyperclusters

Def 17. *A* **hypercluster** σ_j *of a hypergraph* $\mathfrak{H}$ *is a maximal subhypergraph which cannot be divided into two subhypergraphs sharing only one common hypervertex.*



$\sigma \subset \mathfrak{H}$

Every hypercycle determines a unique hypercluster (to which it belongs), while hypercluster may contain several hypercycles.

Factorisation of metric graphs

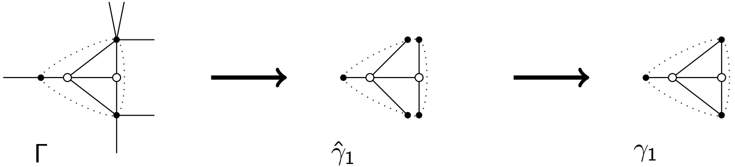
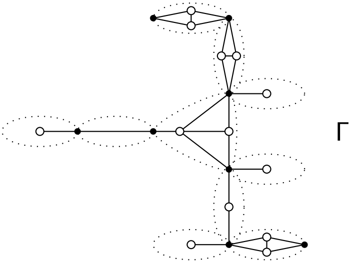


Figure: Isolation of a factor γ_1 from Γ: distinguished vertices each denoted by •; all other vertices are denoted by ○. The distinguished vertices are totally dissolved. Given a resulting connected component $\hat{\gamma}_1$, its endpoints which were identified in Γ are then re-identified to form the factor γ_1 of Γ.



The second reduction: Hypercore

Fix $\lambda \in \mathbb{R}$.
λ is not an eigenvalue of $L^{\mathrm{D}}(\mathfrak{e}_n) \Rightarrow$ the factor is **redundant**.
$\mathfrak{e}_n$ has a contact point belonging to degree one hypervertex

- no eigenfunction on $\mathfrak{e}_n$ satisfies Kirchhoff condition
  $\Rightarrow$ the factor is **redundant**
- otherwise consider **only** eigenfunctions satisfying Dir and Kirch conditions.

Def 18. *Deleting degree one hypervertices, redundant hyperedges and redundant connections in the original hypergraph $\mathfrak{H} = \Gamma_0$, the new graph may again have degree one hypervertices and redundant hyperedges and contacts. The hypergraph formed by repeating this procedure until it stabilises is called the* **Hermitian hypercore of $\mathfrak{H}$ (for L)**, *and is denoted by $\mathfrak{H}_0 = \mathfrak{H}_0(\lambda)$.*

Degree one hypervertices are removed, but only functions satisfying both Dirichlet and Kirchhoff conditions (at the removed hypervertices) are allowed.
OUR ANALYSIS REDUCES TO THE HERMITIAN HYPERPCORE!!

Hypertrees

Thm 19. $\mathfrak{K}$ – *any subhypergraph of the Hermitian hypercore* $\mathfrak{H}_0$, *Statement:*

(S) λ *is a common eigenvalue of the operators* $L^{\mathrm{D}}_{\partial\mathfrak{K}}(\gamma_j)$ *on the hyperedges of* $\mathfrak{K}$

$$\lambda \in \Sigma(L^{\mathrm{D}}_{\partial\mathfrak{K}}(\gamma_j)), \quad \gamma_j \in \mathfrak{K}.$$

Then we have the following:

(a) *If L is* <u>*not*</u> *completely non-self-adjoint, then there exists a non-trivial connected subhypergraph* $\mathfrak{K}$ *of* $\mathfrak{H}_0$ *such that (S) holds.*

(b) *If there exists a non-trivial sub-hypergraph* $\mathfrak{K}$ *which is a also a hypertree, such that (S) holds, then L is* <u>*not*</u> *completely non-self-adjoint.*

(c) *If* $\mathfrak{H}_0$ *is a hypertree, then L is* <u>*not*</u> *completely non-self-adjoint* **iff** *there exists* $\mathfrak{K}$ *such that (S) holds.*

On hypertress eigenfunctions on the factors can always be connected together leading to an eigenfunction.

Hypercycles

Lemma 20.
$\mathfrak{K}$ – subhypergraph which is a hypercycle.
ψ_j - eigenfunctions on the factors enumerated cyclically.
If

$$\prod_{j=1}^{n} \partial\psi_{j+1}(\mathfrak{v}_j) = (-1)^n \prod_{j=1}^{n} \partial\psi_j(\mathfrak{v}_j),$$

holds $\Rightarrow \exists$ eigenfunction supported on $\mathfrak{K}$.

Thm 21. *All hypervertices in the Hermitian hypercore $\mathfrak{H}_0$ have degree at most two. Statements:*

- **(S)** *λ is a common eigenvalue of the operators $L^{\mathrm{D}}_{\partial\mathfrak{K}}(\gamma_j)$ on the hyperedges*
- **(M)** *the multiplicity of the eigenvalue λ on each factor is 1;*
- **(B)** *the eigenfunctions are balanced along any hypercycle in $\mathfrak{K}$.*

Then:

- **(a)** *L is <u>not</u> completely non-self-adjoint $\Leftrightarrow \exists$ subhypergraph $\mathfrak{K}$ of $\mathfrak{H}_0$ and a set of eigenfunctions ψ_j satisfying (S) and (B).*
- **(b)** *If (M) holds for all factors in $\mathfrak{H}_0$, $\Rightarrow$ L is <u>not</u> completely non-self-adjoint if and only if there exists $\mathfrak{K}$ such that (S) and (B) hold.*

General hypertrees

Def 22. *A* **multipath** *in a hypertree $\mathfrak{H}$ is a subhypertree connecting two or more terminals in $\mathfrak{H}$ such that every hypervertex has hyperdegree 2.*

Multipath dimension – the number of independent multipaths.

Thm 23. *Hermitian hypercore $\mathfrak{H}_0(\lambda)$ is a subhyperforest. Then the multiplicity is*

$$m(\lambda) = -\beta_0(\mathfrak{H}_0) + \sum_{\gamma_j} (1 + m_{\gamma_j} - \mathrm{val}_{\mathfrak{H}_0}(\gamma_j)),$$

where

- m_{γ_j} *– multiplicity of λ on the factors;*
- $\beta_0(\mathfrak{H}_0)$ *is the number of connected components of $\mathfrak{H}_0$.*

Every subhypertree gives an eigenfunction!

General hypergraphs

Thm 24. *The multiplicity $m(\lambda)$ satisfies*

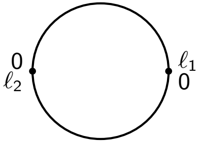
$$m(\lambda) \leq 2\beta_1(\mathfrak{H}_0) - \beta_0(\mathfrak{H}_0) + \sum_{\gamma_j \in \mathfrak{H}_0} \Big(1 + m_{\gamma_j} - \mathrm{val}_{\mathfrak{H}_0}(\gamma_j) \Big)$$

where

- m_{γ_j} – *multiplicity of λ on the factors;*
- $\beta_0(\mathfrak{H}_0)$ – *number of connected components of $\mathfrak{H}_0$,*
- $\beta_1(\mathfrak{H}_0)$ – *number of independent hypercycles in $\mathfrak{H}_0$.*

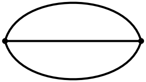
Examples

1. Cycle formed by two edges:



Real eigenvalues iff the edge lengths are rationally dependent.

2. Watermellon



Real eigenvalues if and only if there exists a pair of edge lengths which is rationally dependent.

Examples

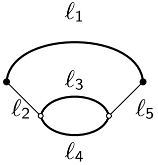


Figure: The graph Γ and the corresponding hypergraph.

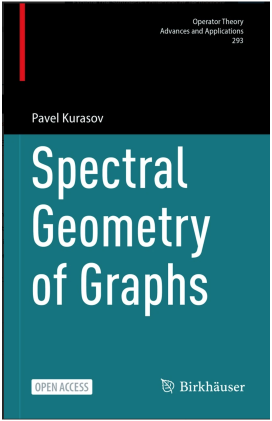
If all edges have equal length ℓ, then the real eigenvalues are $(n\pi/\ell)^2,\ n \geq 1$ with multiplicity two.

Our general estimate:

$$m(\lambda) \leq 2\underbrace{\beta_1(\mathfrak{H}_0)}_{=1} - \underbrace{\beta_0(\mathfrak{H}_0)}_{=1} + \underbrace{\sum_{\gamma_j \in \mathfrak{H}_0} \Big(1 + m_{\gamma_j} - \mathrm{val}_{\mathfrak{H}_0}(\gamma_j)\Big)}_{0+1} = 2$$

The estimate applied to the metric graph:

$$m(\lambda) \leq 2\underbrace{\beta_1(\mathfrak{H}_0)}_{=2} - \underbrace{\beta_0(\mathfrak{H}_0)}_{=1} + \underbrace{\sum_{\gamma_j \in \mathfrak{H}_0} \Big(1 + m_{\gamma_j} - \mathrm{val}_{\mathfrak{H}_0}(\gamma_j)\Big)}_{0+0+0+0+0} = 3$$



Thank you for your attention!