Analytic representations for solutions in coefficient inverse problems

Vladislav V. Kravchenko

Department of Mathematics, Center for Research and Advanced Studies of the National Polytechnic Institute, Mexico

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- - direct and inverse scattering on the line, KdV equation
- - inverse problems on quantum graphs

How to solve linear second order differential equations

$$-y'' + q(x)y = \rho^2 y, \qquad q \in \mathcal{L}_1(0, L), \quad \rho \in \mathbb{C},$$

q - complex valued.

Notation

$$-y'' + q(x)y = \rho^2 y, \qquad q \in \mathcal{L}_1(0, L), \quad \rho \in \mathbb{C}.$$

Notation Solutions $C(\rho, x)$ and $S(\rho, x)$ satisfying

$$C(\rho, 0) = 1, \quad C'(\rho, 0) = h,$$

$$S(\rho, 0) = 0, \quad S'(\rho, 0) = 1.$$

Def.

Any series of the type

$$\sum_{n=0}^{\infty} a_n J_{\nu+n}(z)$$

is called a *Neumann series*, although in fact Neumann considered* only the special type of series for which ν is an integer; the investigation of the more general series is due to Gegenbauer†.

G. N. Watson A treatise on the theory of Bessel functions, 1922.

J. E. Wilkins, Neumann series of Bessel functions, Trans. Amer. Math. Soc. (1948).

A. Baricz, D. Jankov, T. K. Pogány, Series of Bessel and Kummer-type functions. Lect. Notes in Math., Springer, 2017.

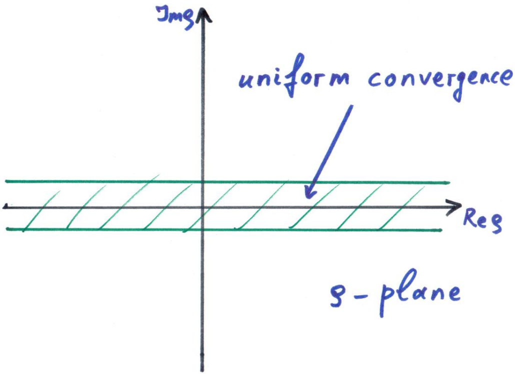


Figure: NSBF representations for solutions converge uniformly in a strip $|\mathrm{Im}\,\rho| \leq c$.

Transmutation (transformation) operators

There exist $\mathbf{C}(x,t), \mathbf{S}(x,t)$-continuous, such that

$$C(\rho,x) = \cos(\rho x) + \int_0^x \mathbf{C}(x,t)\cos(\rho t)\; dt, \quad \forall \rho \in \mathbb{C},$$

$$S(\rho,x) = \frac{\sin(\rho x)}{\rho} + \int_0^x \mathbf{S}(x,t)\frac{\sin(\rho t)}{\rho}\, dt, \quad \forall \rho \in \mathbb{C}.$$

[A. Ya. Povzner 1948; B. M. Levitan, *Inverse Sturm-Liouville problems*, VSP, Zeist, 1987; V. A. Marchenko, *Sturm-Liouville Operators and Applications*, AMS Chelsea Publishing, 2011]

Transmutation kernels constructed

V. Kravchenko, L. J. Navarro, S. M. Torba, *Appl. Math. and Comp.*, (2017)

$$\mathbf{C}(x,t)=\sum_{n=0}^{\infty}\frac{g_n(x)}{x}P_{2n}\left(\frac{t}{x}\right),\quad 0<t\le x,$$

where

$$P_k \text{ - Legendre polynomials}$$

$$g_n(x)=\int_0^x c_n(s)g_{n-1}(s)ds,$$

c_n are known,

$$g_0(x)=C(0,x)-1.$$

Analogously,

$$\mathbf{S}(x,t)=\sum_{n=0}^{\infty}\frac{s_n(x)}{x}P_{2n+1}\left(\frac{t}{x}\right),\quad 0<t\le x,$$

$$\mathbf{C}(x,t)=\sum_{n=0}^{\infty}\frac{g_n(x)}{x}P_{2n}\left(\frac{t}{x}\right),\quad \mathbf{S}(x,t)=\sum_{n=0}^{\infty}\frac{s_n(x)}{x}P_{2n+1}\left(\frac{t}{x}\right),$$

$$\Downarrow \qquad \Downarrow$$

$C(\rho,x)=\cos\rho x+\int_0^x \mathbf{C}(x,t)\cos\rho t\,dt,$

$$S(\rho,x)=\frac{\sin\rho x}{\rho}+\int_0^x \mathbf{S}(x,t)\frac{\sin\rho t}{\rho}\,dt$$

$$\Downarrow \quad \Downarrow$$

$$C(\rho,x)=\cos(\rho x)+\sum_{n=0}^{\infty}(-1)^n g_n(x)\mathbf{j}_{2n}(\rho x),$$

$$S(\rho,x)=\frac{\sin(\rho x)}{\rho}+\frac{1}{\rho}\sum_{n=0}^{\infty}(-1)^n s_n(x)\mathbf{j}_{2n+1}(\rho x),$$

where $\mathbf{j}_k(z)$ - spherical Bessel function ($\mathbf{j}_k(z):=\sqrt{\frac{\pi}{2z}}J_{k+\frac{1}{2}}(z)$).

Theorem [Kravchenko, Navarro, Torba 2017] The solutions $C(\rho, x)$ and $S(\rho, x)$ admit the series representations

$$C(\rho, x) = \cos(\rho x) + \sum_{n=0}^{\infty} (-1)^n g_n(x) \mathbf{j}_{2n}(\rho x),$$

$$S(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{\infty} (-1)^n s_n(x) \mathbf{j}_{2n+1}(\rho x).$$

For every $\rho \in \mathbb{C}$ the series converge pointwise. For every $x \in [0, L]$ the series converge uniformly on any compact set of the complex plane of the variable ρ.

Two important features

1. Since

$$g_0(x) = C(0,x) - 1, \quad s_0(x) = 3\left(\frac{S(0,x)}{x} - 1\right),$$

$q(x)$ can be recovered from the first coefficients of the series:

$$q(x) = \frac{g_0''(x)}{g_0(x) + 1}$$

and

$$q(x) = \frac{(xs_0(x))''}{xs_0(x) + 3x}.$$

Two important features

2. Consider the partial sum

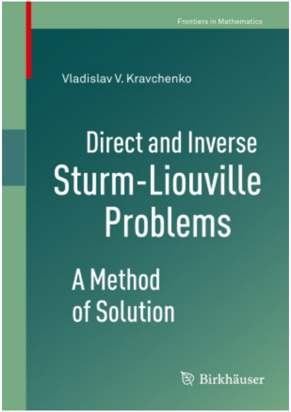
$$S_N(\rho, x) := \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^{N} (-1)^n s_n(x) \mathbf{j}_{2n+1}(\rho x).$$

The estimate holds

$$|\rho S(\rho, x) - \rho S_N(\rho, x)| < \varepsilon_N(x) \quad \text{for all } \rho \text{ from a strip } |\mathrm{Im}\,\rho| \le c \ ,$$

where $\varepsilon_N(x) > 0$ and $\varepsilon_N(x) \to 0$, $N \to \infty$.
Same for $C(\rho, x)$:

$$|C(\rho, x) - C_N(\rho, x)| < \varepsilon_N(x).$$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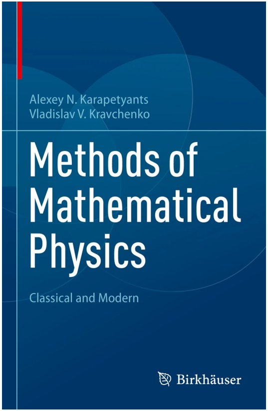
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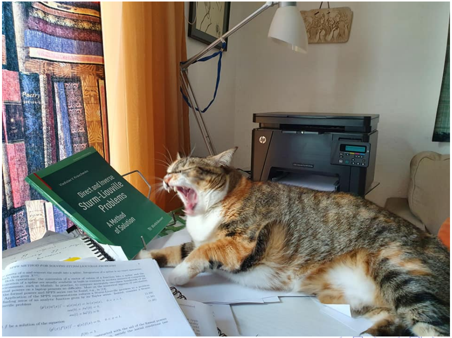
- 
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NSBF are available for

1.

$$-\left(p(x)y'(x)\right)' + q(x)y(x) = \lambda r(x)y(x)$$

[V. V. Kravchenko, S. M. Torba A Neumann series of Bessel functions representation for solutions of Sturm-Liouville equations. Calcolo, 55 (2018)]

2.

$$-y'' + \left(\frac{\ell(\ell+1)}{x^2} + q(x)\right) y = \lambda y,$$

[V. V. Kravchenko, S. M. Torba Transmutation operators and a new representation for solutions of perturbed Bessel equations. Mathematical Methods in the Applied Sciences, 44 (2021)]

3.

$$-y'' + \left(q(x) + \sum_{k=1}^{N} \alpha_k \delta(x - x_k)\right) y = \lambda y,$$

[V. V. Kravchenko, V. A. Vicente-Benitez Closed form solution and transmutation operators for Schrödinger equations with finitely many δ-interactions. Analysis and Mathematical Physics, 14 (2024)]

Better NSBF for nice potentials

Theorem Let $q \in \mathcal{H}^1(0, L)$. Then

$$C(\rho, x) = \cos(\rho x) + \omega(x)\frac{\sin(\rho x)}{\rho} - q^-(x)\frac{x\mathbf{j}_1(\rho x)}{\rho} - \frac{1}{\rho^2}\sum_{n=1}^{\infty}\varphi_n(x)\mathbf{j}_{2n}(\rho x),$$

$$S(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{\omega(x)}{\rho^2}\left(\frac{3\mathbf{j}_1(\rho x)}{\rho x} - \cos(\rho x)\right) + \frac{q^+(x)}{\rho^3}\left(\sin(\rho x) - 3\mathbf{j}_1(\rho x)\right) - \frac{1}{\rho^3}\sum_{n=1}^{\infty}\sigma_n(x)\mathbf{j}_{2n+1}(\rho x),$$

$$\omega(x) := \frac{1}{2}\int_0^x q(s)ds, \quad q^{\pm}(x) := \frac{q(x) \pm q(0)}{4} - \frac{\omega^2(x)}{2}.$$

$$q(x) = 2(q^+(x) + q^-(x) + \omega^2(x)).$$

For all ρ from a strip $|\mathrm{Im}\,\rho| \leq c$, $\rho \neq \{0\}$ the remainders of the partial sums admit

$$|C(\rho,x) - C_N(\rho,x)| \leq \frac{\varepsilon_N(x)}{|\rho|^2},$$

$$|S(\rho,x) - S_N(\rho,x)| \leq \frac{\varepsilon_N(x)}{|\rho|^3},$$

where $\varepsilon_N(x)$ is positive and tending to zero when $N \to \infty$. For $\rho = 0$:

$$C(0,x) = C_1(0,x), \quad S(0,x) = S_1(0,x)$$

Obtained in [V. Kravchenko, *Reconstruction techniques for complex potentials*. J Math Phys (2024)] using [V. Kravchenko, S. Torba, *Asymptotics with respect to the spectral parameter and Neumann series of Bessel functions for solutions of the one-dimensional Schrödinger equation*. J Math Phys (2017)]

Transmutation operators with boundary condition at infinity

$$-y'' + q(x)y = \rho^2 y, \quad x > 0, \tag{1}$$

$$(1+x)\, q(x) \in \mathcal{L}_1(0,\infty). \tag{2}$$

$\exists!$ **Jost** solution $y = e(\rho, x)$ such that

$$e(\rho, x) = e^{i\rho x}\left(1 + o(1)\right), \quad x \to \infty. \tag{3}$$

Levin representation:

$$e(\rho, x) = e^{i\rho x} + \int_x^\infty A(x,t) e^{i\rho t}\, dt \tag{4}$$

where $A(x, \cdot) \in \mathcal{L}_2\,(x, \infty)$ [B. Ya. Levin 1956; B. M. Levitan, *Inverse Sturm-Liouville problems*, VSP, Zeist, 1987; Kh. Chadan, P. C. Sabatier, *Inverse problems in quantum scattering theory.* Springer, 1989].

Fourier-Laguerre expansion of the transmutation kernel

[V. Kravchenko, On a method for solving the inverse scattering problem on the line. *Math Meth Appl Sci* (2019)]

$$A(x,t)=\sum_{n=0}^{\infty}a_n(x)L_n(t-x)e^{\frac{x-t}{2}} \qquad L_n \text{ - Laguerre polynomials}$$

$$\Downarrow$$

$$e(\rho,x)=e^{i\rho x}\left(1+\sum_{n=0}^{\infty}a_n(x)\int_0^{\infty}L_n(t)e^{-\left(\frac{1}{2}-i\rho\right)t}dt\right).$$

$$\Downarrow$$

Representation for Jost solution

$$e(\rho, x) = e^{i\rho x}\left(1 + (z+1)\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)\right)$$

where

$$z := \frac{\frac{1}{2} + i\rho}{\frac{1}{2} - i\rho}.$$

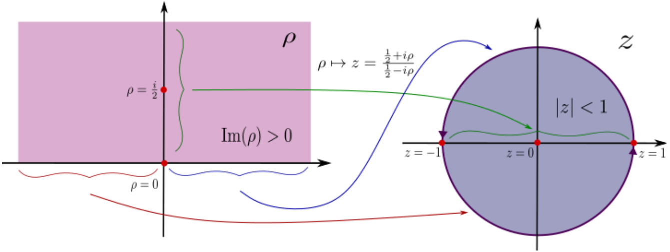

Figure: Möbius mapping of the halfplane $\mathrm{Im}\,\rho \geq 0$ onto the unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

Substitution of this series into the S-L equation leads to a simple recurrent integration procedure for calculating a_n, starting with

$$a_0(x) = e(\frac{i}{2}, x)e^{\frac{x}{2}} - 1.$$

$$a_n(x) \longrightarrow 0 \text{ when } x \longrightarrow \infty, \quad n = 0, 1, \ldots.$$

[B. Delgado, K. Khmelnytskaya, V. Kravchenko, *Math Meth Appl Sci* 2020]
Everything is also valid for complex potentials
[V. Kravchenko, L. E. Murcia-Lozano, *An approach to solving direct and inverse scattering problems for non-selfadjoint Schrödinger operators on a half-line*. Mathematics 2023, 11, 3544]

- $e(\rho, x)e^{-i\rho x}$ is a power series in terms of z.

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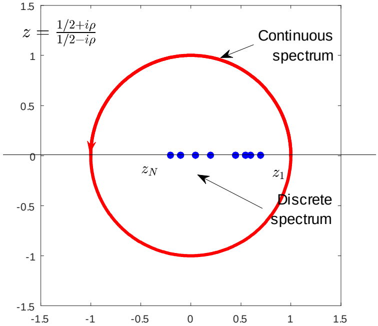


Truncation error estimate

$$e_N(\rho,x) := e^{i\rho x}\left(1+(z+1)\sum_{n=0}^{N}(-1)^n z^n a_n(x)\right).$$

Theorem 1) Let $\operatorname{Im}\rho > 0$. Then

$$\left|e(\rho,x)-e_N(\rho,x)\right| \le \varepsilon_N(x)\frac{e^{-\operatorname{Im}\rho\, x}}{\sqrt{2\operatorname{Im}\rho}} \tag{5}$$

where

$$\varepsilon_N(x) := \left(\sum_{N+1}^{\infty}\left|a_n(x)\right|^2\right)^{1/2} = \left(\int_0^\infty e^{-t}\left|a(x,t)-a_N(x,t)\right|^2 dt\right)^{1/2} \tag{6}$$

with $a_N(x,t) := \sum_{n=0}^{N} a_n(x)L_n(t)$.
2) For $\rho\in\mathbb{R}$ the equality holds

$$\left\|e(\cdot,x)-e_N(\cdot,x)\right\|_{\mathcal{L}_2(-\infty,\infty)} = \sqrt{2\pi}\varepsilon_N(x). \tag{7}$$

Inverse problem

Problem A

Inverse coefficient problem for

$$-y''(x) + q(x)y(x) = \rho^2 y(x), \quad x \in (0, L),$$

$q \in \mathcal{H}^1 [0, L]$ - **complex** valued.
For a set of values $\rho_k \in \mathbb{C}$, the Cauchy data are given

$$a_k = y(\rho_k, 0) \quad \text{and} \quad b_k = y'(\rho_k, 0),$$

together with

$$\ell_k = y(\rho_k, L).$$

Problem A Given input data

$$\{\rho_k, \, a_k, \, b_k, \, \ell_k\},$$

recover $q(x)$.
Note:

$$y(\rho_k, x) = a_k C(\rho_k, x) + b_k S(\rho_k, x) \qquad (h = 0)$$

Special cases include

1. **Recovery from a Weyl function**

Let $\Phi(\rho, x)$ denote a Weyl solution:

$$\Phi'(\rho, 0) = 1, \qquad \Phi(\rho, L) = 0.$$

If ρ^2 is not a Neumann-Dirichlet eigenvalue, $\Phi(\rho, x)$ exists and is unique.

$$M(\rho) := \Phi(\rho, 0) \qquad \text{Weyl function}$$

Frequently studied inverse Sturm-Liouville problem (see, e.g., V. A. Yurko, *Introduction to the theory of inverse spectral problems*, Fizmatlit, Moscow, 2007) consists in recovering $q(x)$ from $M(\rho)$ (given at a set of points). We have

$$a_k = M(\rho_k), \; b_k = 1 \quad \text{and} \quad \ell_k = 0, \qquad k = 1, 2, \ldots .$$

[Gesztesy F, del Rio R & Simon B 1997 Int. Math. Res. Notices 15]
[Horvath M 2005 Ann Math 162]
[Bondarenko N P 2020 Open Math 18]

2. Inverse two-spectrum problem

μ_j^2 - eigenvalues of the Sturm–Liouville problem

$$y'(0) - h_1 y(0) = 0, \quad y(L) = 0, \tag{8}$$

ν_j^2 - eigenvalues of

$$y'(0) - h_2 y(0) = 0, \quad y(L) = 0, \tag{9}$$

$h_{1,2} \in \mathbb{C}$, $h_1 \neq h_2$. Consider a solution $u(\rho, x)$ such that

$$u(\mu_j, 0) = u(\nu_j, 0) = 1, \quad j = 0, 1, \dots,$$

and

$$u'(\mu_j, 0) = h_1, \ u'(\nu_j, 0) = h_2, \quad j = 0, 1, \dots.$$

Thus, $u(\rho, x)$ is an eigenfunction of (8) when $\rho = \mu_j$, and that of (9) when $\rho = \nu_j$. The set of ρ_k is chosen as $\{\mu_0, \nu_0, \mu_1, \nu_1, \cdots\}$. The set of data:

$$a_k = 1, \quad b_k = \begin{cases} h_1, & \text{when } \rho_k = \mu_j, \\ h_2, & \text{when } \rho_k = \nu_j, \end{cases} \quad \text{and} \quad \ell_k = 0, \tag{10}$$

$k = 1, 2, \dots$.

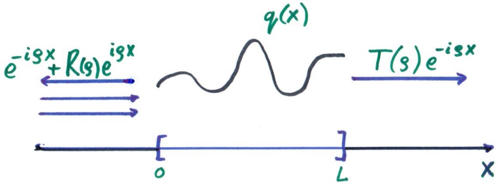


Figure: Plane wave $e^{-i\rho x}$ incoming from $-\infty$ interacts with $q(x)$

The resulting wave is

$$y(\rho,x) = e^{-i\rho x} + R(\rho)e^{i\rho x}, \quad x<0,$$

$$y(\rho,x) = T(\rho)e^{-i\rho x}, \quad x>L,$$

$R(\rho)$ - reflection, $T(\rho)$ - transmission coefficients.

T. Aktosun, Bound states and inverse scattering for the Schrödinger equation in one dimension. *J. Math. Phys.* 35 (1994).

W. Rundell, P. E. Sacks, On the determination of potentials without bound state data. *J. Comp. & Appl. Math.*, 55 (1994).

Problem. On a set of points $\rho_k \in \mathbb{C}$ the values of $R(\rho_k)$ and $T(\rho_k)$ are known, recover $q(x)$.

For the solution $y(\rho_k, x)$ we have the initial conditions

$$y(\rho_k, 0) = 1 + R(\rho_k) =: a_k,$$

$$y'(\rho_k, 0) = -i\rho_k(1 - R(\rho_k)) =: b_k$$

and additionally,

$$\ell_k = T(\rho_k)e^{-i\rho_k L}.$$

Some existing methods for inverse S-L problems

W. Rundell, P. E. Sacks, *Reconstruction techniques for classical inverse Sturm–Liouville problems.* Math. Comput. 58 (1992), 161–83.

M. Ignatiev, V. Yurko, *Numerical methods for solving inverse Sturm-Liouville problems.* Results Math. 52 (2008), no. 1-2, 63–74.

A. Kammanee, C. Böckmann, *Boundary value method for inverse Sturm-Liouville problems.* Appl. Math. & Comp., 214 (2009) 342-352.

B. M. Brown, V. S. Samko, I. W. Knowles, M. Marletta, *Inverse spectral problem for the Sturm–Liouville equation*. Inv. Probl. 19 (2003) 235–252.

B. D. Lowe, M. Pilant, W. Rundell, *The recovery of potentials from finite spectral data*. SIAM J. Math. Anal. 23 (1992), no. 2, 482–504.

None solves Problem A.

None deals with complex potentials.

Many need $\omega(L) := \frac{1}{2}\int_0^L q(s)ds$.

Our previous work

V. Kr., *On a method for solving the inverse Sturm–Liouville problem*, J. Inv. Ill-posed Probl. 27 (2019), 401–407.

V. Kr., S. Torba, *A direct method for solving inverse Sturm-Liouville problems*, Inv. Probl. 37 (2021), 015015.

V. Kr., S. Torba, *A practical method for recovering Sturm-Liouville problems from the Weyl function*. Inv. Probl. 37 (2021), 065011.

V. Kr., **Direct and inverse Sturm-Liouville problems: A method of solution**, Birkhäuser, 2020.

V. Kr., K. V. Khmelnytskaya, F. Çetinkaya, *Recovery of inhomogeneity from output boundary data*. Mathematics, 10 (2022).

V. Kr., V. Vicente-Benitez, *Transmutation operators method for Sturm-Liouville equations in impedance form II: Inverse problem*, J Math Sci, 266 (2022), 554–575.

V. Kr., *Spectrum completion and inverse Sturm-Liouville problems*. Math. Meth. Appl. Sci., 46 (2023), 5821–5835.

F. Çetinkaya, K. Khmelnytskaya, V. Kr., *Neumann series of Bessel functions for inverse coefficient problems*. Math Meth Appl Sci, (2024)

Solution of inverse problem

V. Kravchenko, *Reconstruction techniques for complex potentials*. J Math Phys (2024)

First step

$$\{\rho_k,\ a_k,\ b_k,\ \ell_k\}$$

$$\Downarrow$$

$$C(\rho, L) \quad \text{and} \quad S(\rho, L)$$

$$\text{Given } C(\rho, L) \text{ and } S(\rho, L) \text{, find } q(x).$$

Naturally arises when solving, e.g., two-spectra inverse problems

W. Rundell, P. E. Sacks, *Reconstruction techniques for classical inverse Sturm–Liouville problems.* Math. Comput. 58 (1992).

Kh. Chadan, D. Colton, L. Päivärinta, W. Rundell, *An introduction to inverse scattering and inverse spectral problems*. SIAM, Philadelphia, 1997.

V. Kr., S. M. Torba, *A direct method for solving inverse Sturm-Liouville problems*, Inverse Problems (2021).

or recovering from a Weyl function

V. Kr., S. M. Torba, *A practical method for recovering Sturm-Liouville problems from the Weyl function*. Inverse Problems (2021).

Third solution

$$-T''(\rho,x) + q(x)T(\rho,x) = \rho^2 T(\rho,x), \quad x \in (0,L),$$
$$T(\rho,L) = 0, \quad T'(\rho,L) = 1.$$

Note:

$$T(\rho,x) = C(\rho,L)S(\rho,x) - S(\rho,L)C(\rho,x). \tag{11}$$

This is a generalization of the identity

$$\sin(\rho(x-L)) = \cos(\rho L)\sin(\rho x) - \sin(\rho L)\cos(\rho x).$$

1.

$$\{\rho_k,\, a_k,\, b_k,\, \ell_k\}$$

$$\Downarrow \quad \text{NSBF}$$

$$C(\rho, L) \quad \text{and} \quad S(\rho, L)$$

2.

$$T(\rho, x) = C(\rho, L)S(\rho, x) - S(\rho, L)C(\rho, x)$$

$$\Downarrow \quad \text{NSBF}$$

$$q(x)$$

Solution of Problem A

First step
Substitute NSBF representations of $C(\rho, x)$ and $S(\rho, x)$ into

$$\ell_k = y(\rho_k, L) = a_k C(\rho_k, L) + b_k S(\rho_k, L).$$

$$\Updownarrow$$

$$
\begin{aligned}
&c_1(\rho_k)\,\omega(L) + c_2(\rho_k)\,q^-(L) + c_3(\rho_k)\,q^+(L) \\
&-\frac{a_k}{\rho_k^2}\sum_{n=1}^{\infty}\varphi_n(L)\mathbf{j}_{2n}(\rho_k L) - \frac{b_k}{\rho_k^3}\sum_{n=1}^{\infty}\sigma_n(L)\mathbf{j}_{2n+1}(\rho_k L) \\
&= \ell_k - a_k\cos(\rho_k L) - \frac{b_k}{\rho_k}\sin(\rho_k L),
\end{aligned}
$$

for all ρ_k.
Compute the NSBF coefficients $\omega(L)$, $q^-(L)$, $q^+(L)$, $\varphi_n(L)$, $\sigma_n(L)$, $n = 1, 2, \ldots$.

$$\{\rho_k,\, a_k,\, b_k,\, \ell_k\}_{k=1}^{K}$$

$$\Downarrow \quad \text{NSBF}$$

$$C_N(\rho, L) \quad \text{and} \quad S_N(\rho, L)$$

Second step: Consider

$$T(\rho, x) = C_N(\rho, L) S(\rho, x) - S_N(\rho, L) C(\rho, x)$$

$$\Downarrow \quad \text{NSBF}$$

$$A_{k1}(x)\omega(x) + A_{k2}(x)Q(x) + A_{k3}(x)\omega_L(x) + A_{k4}(x)q_L^+(x)$$
$$+ \sum_{n=1}^{N} B_{kn}(x)\varphi_n(x) + \sum_{n=1}^{N} C_{kn}(x)\sigma_n(x) + \sum_{n=1}^{N} D_{kn}(x)\theta_n(x) = b_k(x),$$

$k = 1, \ldots, m.$

$$Q(x) := \frac{q(x)}{4} - \frac{\omega^2(x)}{2}, \quad \omega_L(x) := \frac{1}{2}\int_x^L q(s)ds,$$
$$q_L^+(x) := \frac{q(x) + q(L)}{4} - \frac{\omega_L^2(x)}{2}.$$

The potential $q(x)$ is obtained from $Q(x)$ and $\omega(x)$:

$$\boxed{q(x) = 4Q(x) + 2\omega^2(x)}$$

Example

$$q(x) = \frac{10\cos(13x)}{(x+0.1)^2} + \pi e^x \sin(20.23x)\, i, \quad x \in [0,1]$$

with the input data of the following form. 101 points ρ_k are chosen uniformly distributed on the segment $[0.1, 100]$, and $a_k = \sin \rho_k$, $b_k = \cos \rho_k$.

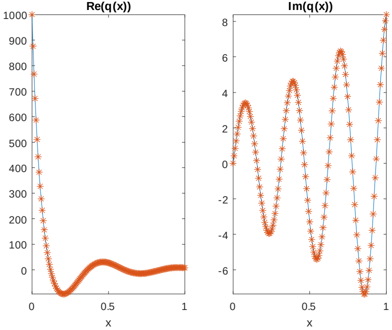


Figure: Max. abs. error $0.8 \cdot 10^{-3}$.

Example

$$q(x) = \int_0^x \left(\left| s - \frac{1}{3} \right| + \pi \left| s - \frac{4}{5} \right| \right) ds + \left(1 - (\pi x - 1)^2 \operatorname{sgn}(1 - \pi x)\right) i.$$

Both $\operatorname{Re} q(x)$ and $\operatorname{Im} q(x)$ have discontinuous second derivatives.
The input data are computed at 101 uniformly distributed random points $\rho_k \in (0, 15)$, with $a_k = \sin \rho_k$, $b_k = \cos \rho_k$.

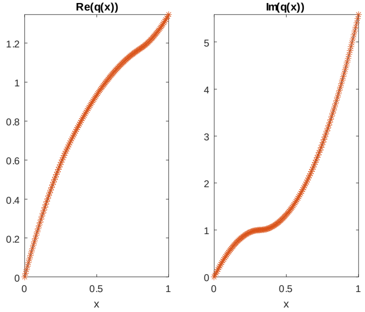


Figure: Max. abs. error $0.2 \cdot 10^{-3}$.

S-L eq. in impedance form, Webster's horn eq.

$$\left(r^2(x)u'(x)\right)' + \rho^2 r^2(x)u(x) = 0, \quad x \in (0, L), \tag{12}$$
$$u'(0) = C, \quad u(L) = 0,$$

$r(x)$ - complex valued from Sobolev class $\mathcal{H}^2(0, L)$, $r(x) \neq 0$, $x \in [0, L]$, $C \neq 0$.

IP1: find $r(x)$, given

$$F(\rho) = u(\rho, 0), \quad \rho \in \Omega, \tag{13}$$

on a set $\Omega \subset \mathbb{C}$.

If $C = 1$, the function $F(\rho)$ is the Neumann-to-Dirichlet map, a.k. as Weyl (or transfer) function. $r_0 = r(0)$ is assumed to be known.

[L. Borcea, V. Druskin, L. Knizhnerman, On the continuum limit of a discrete inverse spectral problem on optimal finite difference grids. *Comm on Pure and Appl Math*, 58 (2005)].

Detecting mechanical imperfections
[A. O. Vatulyan, *Inverse problems of solid mechanics.* Fizmatlit, Moscow, 2007, (in Russian)].

Determining the shape of the human vocal tract from acoustical measurements
[T. Aktosun, P. Sacks, X-Ch. Xu, An inverse problem to determine the shape of a human vocal tract. *J Comp & Appl Math*, 393 (2021)].

Reformulation of IP1

$$-y''(\rho,x)+q(x)y(\rho,x)=\rho^2 y(\rho,x),\quad x\in(0,L). \tag{14}$$

$q\in\mathcal{L}^2\,(0,L)$ - complex valued.

$$y'(\rho,0)-hy(\rho,0)=c,\quad y(\rho,L)=0, \tag{15}$$

$$y(\rho,0)=f(\rho), \tag{16}$$

where h and c are complex constants, and $f(\rho)$ is a function defined on Ω.
IP2: find $q(x)$ and h from $f(\rho)$ and c.
Here

$$q(x)=r''(x)/r(x),$$

$$y(\rho,x)=r(x)u(\rho,x),$$

and

$$f(\rho)=r(0)F(\rho),\quad h=\frac{r'(0)}{r(0)},\quad c=r(0)C.$$

Thus, one may solve IP2 for $f(\rho)$ and c and subsequently calculate $r(x)$, by solving Cauchy's problem

$$r''(x) - q(x)r(x) = 0, \tag{17}$$

$$r(0) = r_0, \quad r'(0) = hr_0. \tag{18}$$

Still a better possibility:

$$r(x) = r_0 C(0, x) = r_0 \left(g_0(x) + 1\right).$$

Solution

$$y(\rho, x) = f(\rho)C(\rho, x) + cS(\rho, x).$$

Condition $y(\rho, L) = 0$ leads to

$$f(\rho)C(\rho, L) + cS(\rho, L) = 0 \tag{19}$$

$$\Downarrow$$

For $\rho_k \in \Omega$:

$$\begin{aligned} & f(\rho_k)\sum_{n=0}^{N}(-1)^n g_n(L)\mathbf{j}_{2n}(\rho_k L) + \frac{c}{\rho_k}\sum_{n=0}^{N}(-1)^n s_n(L)\mathbf{j}_{2n+1}(\rho_k L) \\ = \quad & -f(\rho_k)\cos\rho_k L - c\frac{\sin\rho_k L}{\rho_k}, \quad k = 1, 2, \ldots, K \end{aligned}$$

to find

$$\{g_n(L), s_n(L)\}_{n=0}^{N} \quad \Longrightarrow \quad C_N(\rho, L),\ S_N(\rho, L)$$

$$r(x) = \sqrt{1 + 10e^{-25(x-1/2)^2}}, \qquad x \in [0,1], \; C = -1.$$

[L. Borcea, V. Druskin, L. Knizhnerman, On the continuum limit of a discrete inverse spectral problem on optimal finite difference grids. *Comm on Pure and Appl Math*, 58 (2005)].

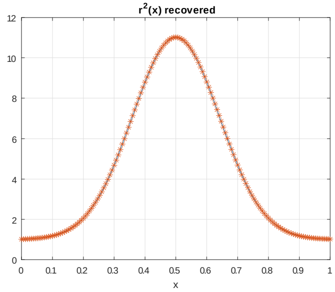


Figure: $r^2(x)$ recovered from $F(\rho)$ given at 20 randomly distributed points ρ_k from the interval $(0.1, 50)$. max abs err $< 2 \cdot 10^{-2}$; max rel err $< 5 \cdot 10^{-3}$.

In the first system: $N = 6$. Thus, $\varphi_6(\rho, L)$ and $S_6(\rho, L)$ were computed.

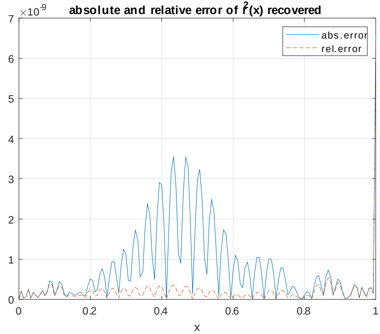


Figure: abs. and rel. errors of $r^2(x)$ recovered from $F(\rho)$ given at 150 randomly distributed points ρ_k from the interval $(0.1, 150)$.

Complex r(x)

$$r(x) = \sqrt{1 + 10e^{-25(x-1/2)^2}} + \frac{i}{(0.1+x)^2}.$$

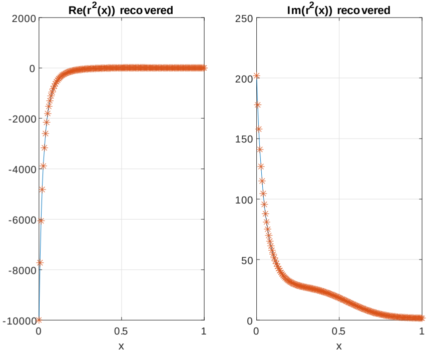


Figure: $r^2(x)$ recovered from $F(\rho)$ given at 150 randomly distributed points ρ_k from the interval $(0.1, 150)$.

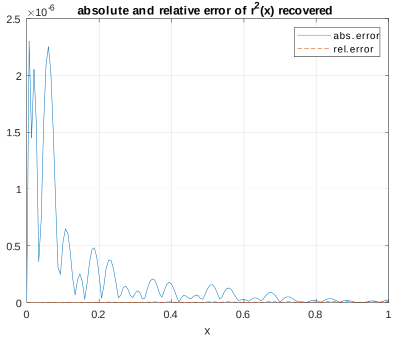


Figure: Abs. and rel. errors of $r^2(x)$ recovered from $F(\rho)$ given at 150 randomly distributed points ρ_k from the interval $(0.1, 150)$.

Oscillating r(x)

$$r(x) = 10e^{\pi^2 ix}.$$

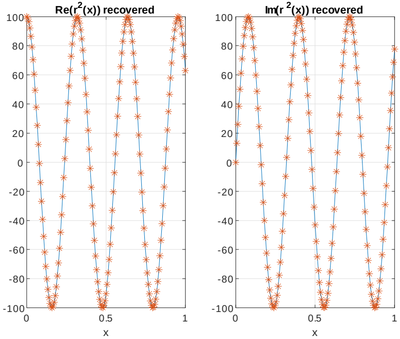


Figure: $r^2(x)$ recovered from $F(\rho)$ given at 150 randomly distributed points ρ_k from the interval $(0.1, 150)$.

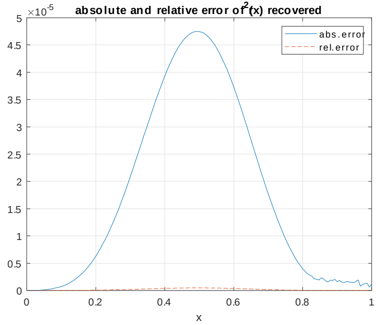


Figure: Abs. and rel. errors of $r^2(x)$ recovered from $F(\rho)$ given at 150 randomly distributed points ρ_k from the interval $(0.1, 150)$. Maximums of abs. and rel. errors are $4.7 \cdot 10^{-5}$ and $4.7 \cdot 10^{-7}$, respectively.

Scattering problem on the line

$$-y'' + q(x)y = \lambda y, \quad x \in (-\infty, \infty)$$

$q(x)$ is real valued, defined on $(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} (1 + |x|) |q(x)| \, dx < \infty. \tag{20}$$

$\lambda = \rho^2$ and $\tau := \operatorname{Im} \rho \geq 0$.
Jost solutions

$$e(\rho, x) = e^{i\rho x} (1 + o(1)), \quad x \to +\infty$$

$$g(\rho, x) = e^{-i\rho x} (1 + o(1)), \quad x \to -\infty$$

[K. Chadan, P. C. Sabatier Inverse problems in quantum scattering theory. Springer, NY, 1989]
[V. A. Yurko, Introduction to the theory of inverse spectral problems. Fizmatlit, Moscow, 2007, (in Russian)]

When $\operatorname{Im}\rho > 0$:

$$e(\rho, x) \in L_2(\alpha, \infty), \quad g(\rho, x) \in L_2(-\infty, \alpha) \quad \forall \alpha \in \mathbf{R}.$$

If for some specific $\rho = \rho_k$:

$$e(\rho_k, x) \in L_2(-\infty, \infty) \quad \text{or} \quad g(\rho_k, x) \in L_2(-\infty, \infty)$$

then

$$e(\rho_k, x) = c_k g(\rho_k, x) \quad \text{- } \textbf{eigenfunction}.$$

The **eigenvalues**, if they exist, form a finite set of negative numbers $\lambda_k = \rho_k^2 = (i\tau_k)^2$, $0 < \tau_1 < \ldots < \tau_N$.

Norming constants

$$\alpha_k^+ := \left(\int_{-\infty}^{\infty} e^2(\rho_k, x) dx \right)^{-1} \quad \text{and} \quad \alpha_k^- := \left(\int_{-\infty}^{\infty} g^2(\rho_k, x) dx \right)^{-1}.$$

Discrete scattering data

$$\left\{ \lambda_k, \alpha_k^+, k = \overline{1, N} \right\} \quad \text{or} \quad \left\{ \lambda_k, \alpha_k^-, k = \overline{1, N} \right\}.$$

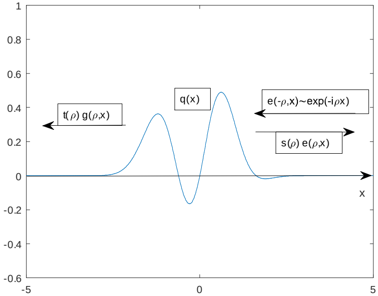
Scattering amplitudes

$$a(\rho) = -\frac{1}{2i\rho} W\left[e(\rho,x), g(\rho,x)\right], \quad b(\rho) = \frac{1}{2i\rho} W\left[e(\rho,x), g(-\rho,x)\right].$$

W - Wronskian.

Reflection coefficients (right and left):

$$s^{\pm}(\rho) := \mp \frac{b(\mp\rho)}{a(\rho)}$$



Right and left scattering data

$$J^{\pm} = \left\{ s^{\pm}(\rho),\, \rho \in \mathbf{R};\, \lambda_k,\, \alpha_k^{\pm},\, k = \overline{1, N} \right\}$$

Direct scattering problem

Given the potential $q(x)$, find the scattering data J^{+} or J^{-}.

Inverse scattering problem

Given the scattering data J^{+} or J^{-}, find the potential q.

Applications

Inverse scattering transform method; nonlinear Fourier transform

Korteweg - de Vries equation

$$u_t + u_{xxx} + 6uu_x = 0, \quad u(x,0) = q(x)$$

schematic representation of the IST

$$u(x,0) = q(x) \quad \xrightarrow[\text{direct scattering problem}]{-y''+q(x)y=\lambda y} \quad J^+(t=0) \quad (\text{or } J^-)$$

$$\Vert$$

linear evolution law

$$\lambda_k(t) = \lambda_k,$$

$$\alpha_k^+(t) = \alpha_k^+ e^{8\tau_k^3 t}$$

$$s^+(\rho, t) = s^+(\rho)\, e^{8i\rho^3 t}$$

$$\Downarrow$$

$$u(x,t) \quad \xleftarrow{\text{inverse scattering problem}} \quad J^+(t)$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Method for solving the Korteweg-de Vries equation. Phys. Rev. Lett., 19, (1967), 1095–1097.

Direct scattering

$$e(\rho,x)=e^{i\rho x}\left(1+(z+1)\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)\right) \tag{21}$$

and

$$g(\rho,x)=e^{-i\rho x}\left(1+(z+1)\sum_{n=0}^{\infty}(-1)^n z^n b_n(x)\right) \tag{22}$$

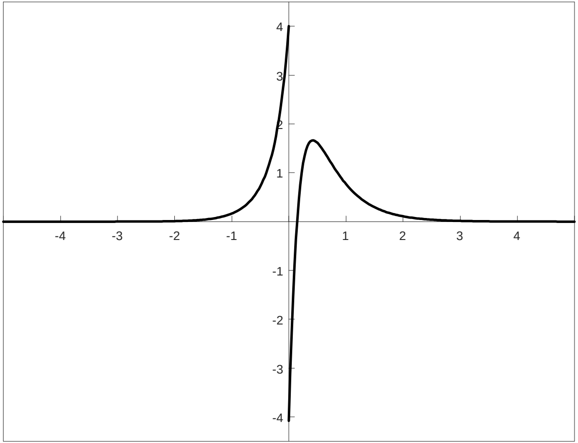
$$\Downarrow$$

scattering data

$$J=\left\{s(\rho),\ \rho\in\mathbf{R};\ \lambda_k,\ \alpha_k,\ k=\overline{1,N}\right\}$$

Example

[T. Aktosun, P. Sacks, Math. Meth. Appl. Sci. 25, (2002), 347-355.]



$$q(x) = \begin{cases} q_1(x), & x < 0 \\ q_2(x), & x > 0 \end{cases}$$

where

$$q_1(x) = \frac{16\left(\sqrt{2}+1\right)^2 e^{-2\sqrt{2}x}}{\left(\left(\sqrt{2}+1\right)^2 e^{-2\sqrt{2}x} - 1\right)^2}$$

and

$$q_2(x) = \frac{96e^{2x}\left(81e^{8x} - 144e^{6x} + 54e^{4x} - 9e^{2x} + 1\right)}{\left(36e^{6x} - 27e^{4x} + 12e^{2x} - 1\right)^2}.$$

Reflection coefficient

$$s(\rho)=\frac{(\rho+i)(\rho+2i)(101\rho^2-3i\rho-400)}{(\rho-i)(\rho-2i)(50\rho^4+280i\rho^3-609\rho^2-653i\rho+400)}.$$

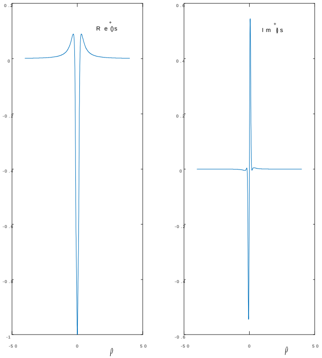


Figure: $s(\rho)$ computed. Abs. error $< 10^{-12}$.

Inverse scattering

$$A(x,t)=\sum_{n=0}^{\infty}a_n(x)L_n(t-x)e^{\frac{x-t}{2}}$$

$$\swarrow \searrow$$

$$F(x+y)+A(x,y)+\int_x^{\infty}A(x,t)F(t+y)dt=0 \quad \text{GLM equation}$$

$F(x)=\sum_{k=1}^{N}\alpha_k e^{-\tau_k x}+\frac{1}{2\pi}\int_{-\infty}^{\infty}s(\rho)\,e^{i\rho x}d\rho.$

$$\Downarrow$$

$$a_m(x)+\sum_{n=0}^{\infty}a_n(x)A_{mn}(x)=r_m(x),\quad m=0,1,\ldots$$

$$\Downarrow$$

$$q=\frac{a_0''-a_0'}{a_0+1}$$

V. Kr. On a method for solving the inverse scattering problem on the line. *Math Meth Appl Sci* (2019)

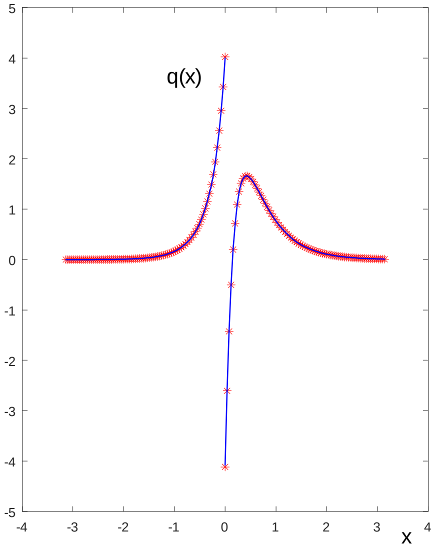
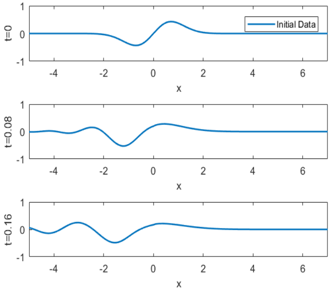


Figure: Potential $q(x)$ recovered with 25 equations.

Example [G. L. Lamb Elements of soliton theory. John Wiley & Sons, 1980]

$$u(x,0) = xe^{-x^2}.$$



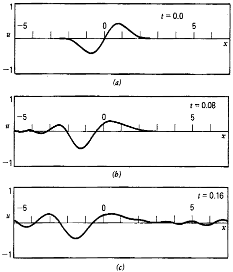


Figure 4.3 Nonsoliton contribution to solution of Korteweg–deVries equation $u_t - 6uu_x + u_{xxx} = 0$ for initial profile $u(x,0) = xe^{-x^2}$. Propagation is from left to right. Precursor is due to imposition of periodic boundary conditions. (Solution provided by W. E. Ferguson, Jr.).

[S. Grudsky, V. Kr., S. Torba Realization of the inverse scattering transform method for the Korteweg-de Vries equation. *Math Meth Appl Sci* (2023)]

Example Cauchy problem for KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

with the initial data

$$u(x,0) = q(x) := \begin{cases} e^x \cos 4x, & x < 0, \\ e^{-x} J_0(2x), & x > 0, \end{cases} \tag{23}$$

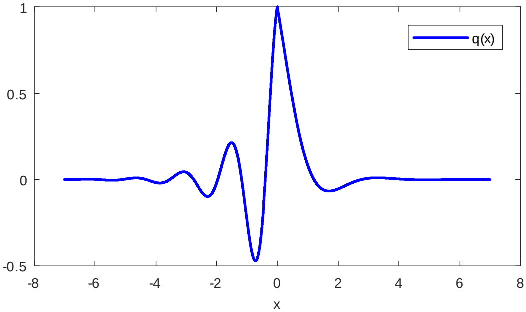


Figure: First derivative of $q(x)$ is discontinuous.

Example

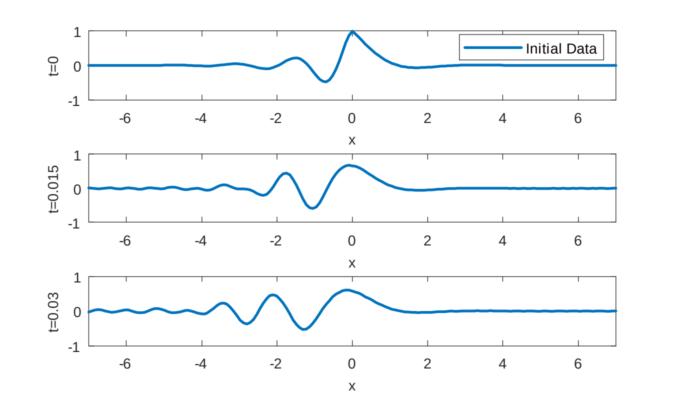


Figure: Solution computed with nine equations in the truncated systems (with five equations it was only slightly less accurate). For $t = 0$ the potential on $(-7, 7)$ was recovered with abs. error of $6 \cdot 10^{-3}$.

Quantum graphs

[S. A. Avdonin, K. V. Khmelnytskaya, V. Kr. *Reconstruction techniques for quantum trees.* Math Meth Appl Sci (2024)] Combined the leaf peeling (Avdonin, Kurasov 2008) with the NSBF approach.

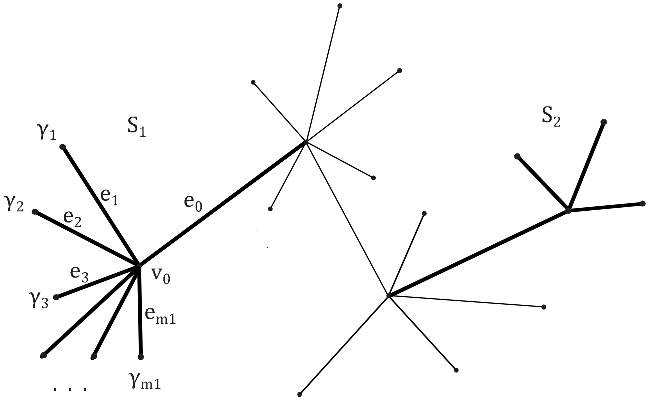


Figure:

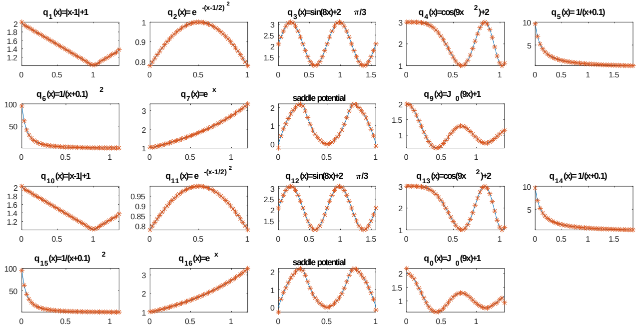


Figure: The potential of the quantum graph, recovered from the Weyl matrix given at 180 points, with $N = 9$.

Thank you!