Schrodinger operators on periodic discrete and metric graphs

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Let $G = (\mathcal{V}, \mathcal{E})$ be a connected infinite graph, possibly having loops and multiple edges and embedded into the space $\mathbb{R}^d$. Here $\mathcal{V}$ is the set of its vertices and $\mathcal{E}$ is the set of its unoriented edges. If the vertices u, v are jointed by some edge, then we denoted it by $u \sim v$. Let Γ be a lattice in $\mathbb{R}^d$ with a basis $a_1, \ldots, a_d$, i.e.,

$$\Gamma = \left\{ a : a = \sum_{s=1}^{d} n_s a_s, \; n_s \in \mathbb{Z}, \; \right\}.$$

We define the equivalence relation on $\mathbb{R}^d$:

$$x = y \pmod{\Gamma} \quad \Leftrightarrow \quad x - y \in \Gamma \qquad \forall\, x, y \in \mathbb{R}^d.$$

We assume that a graph G satisfies the following conditions:

1) $G = G + a$ for any $a \in \Gamma$;
2) the **fundamental** graph $G_0 := G/\Gamma$ is finite.

The basis $a_1, \ldots, a_d$ of the lattice Γ will be called *the periods* of G. The fundamental graph G_0 is a graph on the d-dimensional torus $\mathbb{R}^d/\Gamma$. The fundamental graph $G_0 = (V_0, \mathcal{E}_0)$ has the vertex set $V_0 = V/\Gamma$, the set $\mathcal{E}_0 = \mathcal{E}/\Gamma$ of edges, which are finite.

Let $\ell^2(\mathcal{V})$ be the space of functions $f : \mathcal{V} \to \mathbb{C}$ equipped with the norm $\|f\|^2_{\ell^2(\mathcal{V})} = \sum_{v\in\mathcal{V}} |f(v)|^2 < \infty$. We define the discrete Laplacian Δ on G by

$$\big(\Delta f\big)(v) = \sum_{v\sim u} \big(f(v) - f(u)\big), \qquad f = (f(v)) \in \ell^2(\mathcal{V}), \quad v \in \mathcal{V}.$$

Recall that Δ is self-adjoint and $0 \in \sigma(\Delta) \subset [0, \Lambda_+]$ for some $\Lambda_+ > 0$. The spectrum of the operator Δ on the periodic graph G has the form

$$\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta),$$

where $\sigma_{ac}(\Delta)$ is the absolutely continuous spectrum, which is a union of non-degenerate bands, and $\sigma_{flat}(\Delta)$ is the set of all flat bands (eigenvalues of infinite multiplicity). An open interval between two neighboring non-degenerate bands is called a *gap*.

The Lebesgue measure $|\sigma(\Delta)|$ of the spectrum of the Laplacian Δ satisfies an estimate from EK + Saburova [2014]

$$|\sigma(\Delta)| \leqslant 4\beta,$$

where $\beta = \#\mathcal{E}_0 - \#\mathcal{V}_0 + 1$ is the Betti number of the fundamental graph G_0. Note that the Betti number β can also be defined by the following equivalent ways:

the dimension of the cycle space $\mathcal{C}$ of the graph G_0, i.e., $\beta = \dim \mathcal{C}$.

We consider the Schrödinger operator $H = \Delta - V$ on $\ell^2(\mathcal{V})$.

Theorem (EK + Slouschs, SPb Math. J. 2020). *Consider the Schrödinger operator $H = \Delta - V$ on $\ell^2(\mathcal{V})$, where the potential $V \geqslant 0$ and $V \in \ell^{\frac{d}{2}}(\mathcal{V}), d \geqslant 3$. Then the operator H has a finite number of negative eigenvalues (counting with multiplicity) and their number $\mathcal{N}$ satisfies*

$$\mathcal{N} \leqslant C\|V\|^p_{\ell^p(\mathcal{V})}, \quad p = \frac{d}{2},$$

for some $C = Const(G, d)$. Moreover, if $H = \Delta - \tau V, \tau > 1$, then

$$\mathcal{N} = o(\tau^p) \qquad \text{as} \quad \tau \to +\infty,$$

Remarks. 1) Theorem for $G = \mathbb{Z}^d$ was obtained by Levin, Rozenblum, Solomyak and by Bach.
2) In the proof we use methods of Birman devoted the continuous case $\mathbb{R}^d$. We also need results of Korotyaev-Saburova [2016] about the effective masses for Laplacian on periodic graphs.
3) In the case of complex potentials we do not know when the number of eigenvalues $\notin \sigma_{ess}(H)$ is finite.

We consider Schrödinger operators $H = \Delta + V$ on cubic lattice $\mathbb{Z}^d$, $d \geqslant 3$, where Δ is the Laplacian and the potential $V \in \ell^p(\mathbb{Z}^d)$ for some $p \in [1, \infty)$. We discuss the following:

1) We consider the Laplacian Δ on the lattice and estimate the group $e^{it\Delta}$ and the resolvent $(\Delta - \lambda)^{-1}$ acting in some Banach spaces uniformly with respect to the spectral parameter from the upper half-plane.

2) We apply these results to Schrödinger operators with real decaying potentials from some Banach spaces.

3) We consider the inverse problems for the Schrödinger operators

4) We consider Schrödinger operators with complex decaying potentials from some Banach spaces. In particular, we obtain trace formulas and estimate eigenvalues in terms of the potential.

Consider Schrödinger operators $H = H_0 + V$ on $\ell^2(\mathbb{Z}^d), d \geqslant 3$, where Δ is the discrete Laplacian on $\mathbb{Z}^d$ given by

$$\big(\Delta f\big)(n) = \frac{1}{2} \sum_{|n-m|=1} f_m, \quad n = (n_j)_{j=1}^d \in \mathbb{Z}^d,$$

for $f = (f_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. The potential $V = (V_n)_{n \in \mathbb{Z}^d}$, $V_n \in \mathbb{R}$, is a multiplication operator

$$(Vf)(n) = V_n f_n.$$

We always assume that $V \in \ell^p(\mathbb{Z}^d)$ for some $p \in [1, \infty)$, where $\ell^p(\mathbb{Z}^d)$ is the space of sequences $f = (f_n)_{n \in \mathbb{Z}^d}$ equipped with the norm

$$\|f\|_p^p = \|f\|_{\ell^p(\mathbb{Z}^d)}^p = \sum_{n \in \mathbb{Z}^d} |f_n|^p < \infty.$$

We can describe spectral properties of H by the Fourier transformation $\Phi : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$, defined by

$$\widehat{f}(k) = (\Phi f)(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} f_n e^{-in \cdot k}, \quad k = (k_j)_1^d \in \mathbb{T}^d,$$

where $\mathbb{T}^d = \mathbb{R}^d/(2\pi)^d\mathbb{Z}^d$ is the flat torus. Thus we obtain

$$\widehat{\Delta} = \Phi\Delta\Phi^* = \sum_{j=1}^{d} \cos k_j, \quad \sigma(\Delta) = \sigma_{ac}(\Delta) = [-d, d],$$

$$\widehat{H} = \Phi H \Phi^* = \widehat{\Delta} + \mathcal{V},$$

$$(\mathcal{V}\ \widehat{f})(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} \widehat{V}(k - k')\widehat{f}(k')dk',$$

$$\widehat{V}(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} V_n e^{-in \cdot x}.$$

In fact, this is a special case of the Friedrichs model.

There are a lot of papers and books about dim $d = 1$. Mainly these results are devoted to scattering for the Jacobi operators with decreasing perturbations on the lattice $\mathbb{Z}$. These results are applied to so-called Toda lattice. The case dim $d > 1$ is more complicated. The main problem: we have not a good representation of the Laplacian resolvent.

There are results about spectral properties of discrete Schrödinger operators $H = \Delta + V$ with decreasing potentials on $\mathbb{Z}^d, d \geqslant 2$. We describe these results.

Boutet de Monvel and Sahbani [99] describe the properties of H when $|V_n| \leqslant C(1+|n|)^{-a}, a > 1$:

(1) $\sigma_p(H) \setminus S_d$ is discrete and finite multiplicities with possible accumulation points in $S_d = [-d, d] \cap (d + 2\mathbb{Z})$ where $S_2 = \{0, \pm 2\}$ and $S_3 = \{\pm 1, \pm 3\}$,..... and

$$\sigma_{ac}(H) = \sigma(H_0) = \sigma_{ac}(H_0) = [-d, d], \qquad \sigma_{sc}(H) = \emptyset.$$

(2) Let ρ be a multiplication operator by $\rho_n = \frac{1}{(1+|n|)}$. An operator-valued function

$$F(\lambda) = |V|^{\frac{1}{2}}(\Delta - \lambda)^{-1}|V|^{\frac{1}{2}}, \qquad a > 2,$$

on $\ell^2(\mathbb{Z}^d)$ is analytic in $\mathbb{C} \setminus [-d, d]$ and continues up to the boundary without set of critical points $S_d = [-d, d] \cap (d + 2\mathbb{Z})$,

(3) The wave operators

$$W_\pm = s - \lim\, e^{itH} e^{-itH_0}, \qquad t \to \pm\infty,$$

exist and are complete, i.e. the range of $W_\pm$ is equal to $\mathcal{H}_{ac}(H)$.

Isozaki-Marioka [14] show non-existence of embedded eigenvalues on the interval $(-d, d)$ for $H = \Delta + V$ with finitely supported potentials V in the whole space $\mathbb{Z}^d$.

Rosenblum-Solomjak [09] obtain sharp estimates for the number of negative eigenvalues.

Hundertmark-Simon [02] determine so called Lieb-Thirring type inequalities for the discrete case.

Korotyaev-Kutsenko [10] study the inverse problem for the discrete Schrödinger operators on graphene nano-ribbons in external electric fields.

Recently, Hayashi, Higuchi, Nomura, and Ogurisu computed the number of discrete eigenvalues for finitely supported potential.

Parra and Richard [18] extended the results of Sahbani and Boutet de Monvel to general graphs.

Eigenvalues and trace formulas for discrete Schrödinger operators with complex potentials on $\mathbb{Z}^d, d \geqslant 3$ were described by Korotyaev-Laptev [18], and Malamud-Neidhardt [15].

Inverse problem. Now we discuss the case $d \geqslant 3$ and uniformly decaying potentials V.

Theorem (Isozaki + Korotyaev, Ann. H. Poincare, 2012) *Let $d \geqslant 3$ and let V satisfy*

$$|V(n)| \leqslant \frac{C}{(1+|n|)^a}, \quad \forall n \in \mathbb{Z}^d, \quad a > 2.$$

Then

(i) The operator-valued function $|V|^{\frac{1}{2}}(\Delta - \lambda)^{-1}|V|^{\frac{1}{2}}$ is analytic in the cut domain $\mathbb{C} \setminus [-d, d]$ and is Hölder up to boundary.

(ii) Each eigenvalue of H has a finite multiplicity.

(iii) Suppose a potential V is finitely supported. Then from the scattering operator $S = W_+^* W_-$, one can reconstruct V. Moreover, the mapping $V \to S$ is an injection.

It is a first result about the inverse scattering problem for $\mathbb{Z}^d, d \geqslant 3$. Later on, Ando, Isozaki, ... describe the inverse scattering problems for other graphs.

1. Estimates. In the famous paper Kato [1966] considered the Laplacian Δ on the space $L^2(\mathbb{R}^d), d \geqslant 2$. He proved the following estimates:

(i) For all $t \in \mathbb{R} \setminus \{0\}$ and $u \in L^q(\mathbb{R}^d), q > 2$, we have

$$\|ue^{it\Delta}u\| \leqslant C_d|t|^{-\frac{d}{q}}\|u\|_q^2,$$

for some constant C_d depending on q only.

(ii) Let $u \in L^{d\pm\varepsilon}(\mathbb{R}^d), d > 2+\varepsilon$ for some $\varepsilon > 0$. Then

$$\|u(-\Delta-\lambda)^{-1}u\| \leqslant C_{d,\varepsilon}(\|u\|_{d-\varepsilon}^2 + \|u\|_{d+\varepsilon}^2), \qquad \forall\, \lambda \in \mathbb{C} \setminus [0,\infty),$$

for some constant $C_{d,\varepsilon}$ depending on q, ε only.

These estimates are very important in mathematical physics. There are a lot of papers where these estimates were used.

Our main goal to determine similar estimates for the graphs.

We assume that V satisfies

$$V \in \ell^p(\mathbb{Z}^d), \quad \begin{cases} 1 \leqslant p < \frac{6}{5} & \textit{if} \quad d = 3 \\ 1 \leqslant p < \frac{3d}{1+2d} & \textit{if} \quad d \geqslant 4 \end{cases}. \tag{1}$$

Theorem (EK+Moller, 2020 Ark. Math.). *Let complex $V \in \ell^p(\mathbb{Z}^d), d \geqslant 2, p \geqslant 1$. Then*

$$\||V|^{\frac{1}{2}} e^{it\Delta} |V|^{\frac{1}{2}}\| \leqslant |t|^{-\frac{d}{3p}} \|V\|_p, \qquad \forall\, t \in \mathbb{R} \setminus [-1, 1].$$

Let in addition V satisfy (1) and $d \geqslant 3$ and let
$Y_0(\lambda) = |V|^{\frac{1}{2}} (\Delta - \lambda)^{-1} |V|^{\frac{1}{2}}, \quad \lambda \in \mathbb{C} \setminus [-d, d]$.
Then the operator-valued function $Y_0 : \mathbb{C} \setminus [-d, d] \to \mathcal{B}_2$ (Gilbert-Schmidt class) is analytic and Hölder up to the boundary. Moreover, it satisfies

$$\|Y_0(\lambda)\|_{\mathcal{B}_2} \leqslant C_* \|V\|_p, \qquad \forall\, \lambda \in \mathbb{C} \setminus [-d, d],$$

where C_* depends on p, d only and we have an estimate of C_* in terms of p, d. Moreover, if $C_* \|V\|_p < 1$, then the operator $H = \Delta + V$ does not have eigenvalues.

Remarks. 1) Kato [66] obtained similar results for the continuous case $\mathbb{R}^d$.

2) The method of Kato does not work for $\mathbb{Z}^d$ and conversely our method does not work for $\mathbb{R}^d$.

3) In the proof we use the recent integral estimates of Bessel functions from Krasikov' s paper.

Now we describe spectral properties of Schrödinger operators

Theorem (Korotyaev, Moller, 2020 Ark. Math.). *Let a real potential V satisfy (1) and $d \geqslant 3$. Then the wave operators*

$$W_{\pm} = s - \lim e^{itH} e^{-itH_0} \qquad as \qquad t \to \pm\infty$$

exist and are complete, i.e., $\mathscr{H}_{\rm ac}(H) = \mathrm{Ran}\, W_{\pm}$ and

$$\sigma_{\rm ac}(H) = \sigma_{\rm ac}(\Delta) = [-d, d],$$

2. Complex potentials. We consider the Schrödinger operator $H = \Delta + V$. We assume that the potential V is complex and satisfies the same (1):

$$V \in \ell^p(\mathbb{Z}^d), \quad \begin{cases} 1 \leqslant p < \frac{6}{5} & \textit{if} \quad d = 3 \\ 1 \leqslant p < \frac{3d}{1+2d} & \textit{if} \quad d \geqslant 4 \end{cases}. \qquad (2)$$

Recall that the spectrum of the Laplacian Δ is absolutely continuous and

$$\sigma(\Delta) = \sigma_{\mathrm{ac}}(\Delta) = [-d, d].$$

It is also well known that if V satisfies (2), then the essential spectrum of the Schrödinger operator H on $\ell^2(\mathbb{Z}^d)$ is

$$\sigma_{\mathrm{ess}}(H) = [-d, d].$$

The operator H has $N \leqslant \infty$ eigenvalues $\{\lambda_n, n = 1, ..., N\}, N \leqslant \infty$ outside the interval $[-d, d]$.

Define disc $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\} \subset \mathbb{C}$ with the radius $r > 0$; Define **the new spectral variable** $z \in \mathbb{D} = \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2}\left(z + \frac{1}{z}\right) \in \Lambda = \mathbb{C} \setminus [-d, d], \qquad z \in \mathbb{D} = \mathbb{D}_1.$$

The function $\lambda(z)$ has the following properties:

-) *The function $\lambda(z)$ is a conformal mapping from $\mathbb{D}$ onto the spectral domain Λ, which is the cut domain with the cut $[-d, d]$, having the upper side $[-d, d] + i0$ and the lower side $[-d, d] - i0$.*
-) *$\lambda(\mathbb{D}) = \Lambda = \mathbb{C} \setminus [-d, d]$ and $\lambda(\mathbb{D} \cap \mathbb{C}_{\mp}) = \mathbb{C}_{\pm}$.*
-) *The function $\lambda(z)$ maps the boundary: the upper semi-circle onto the lower side $[-d, d] - i0$ and the lower semi-circle onto the upper side $[-d, d] + i0$.*
-) *The function $\lambda(z)$ maps the point $z = 0$ to the point $\lambda = \infty$.*
-) *The inverse mapping $z(\cdot) : \Lambda \to \mathbb{D}$ is given by*

$$z = \frac{1}{d}\left(\lambda - \sqrt{\lambda^2 - d^2}\right), \qquad \lambda \in \Lambda = \mathbb{C} \setminus [-d, d],$$

$$z = \frac{d}{2\lambda} + \frac{O(1)}{\lambda^3} \qquad as \quad |\lambda| \to \infty.$$

Define the Hardy space $\mathscr{H}_p$. For $0 < p \leqslant \infty$ we say F belongs the Hardy space $\mathscr{H}_p = \mathscr{H}_p(\mathbb{D})$ if F is analytic in $\mathbb{D}$ and satisfies $\|F\|_{\mathscr{H}_p} < \infty$, where $\|F\|_{\mathscr{H}_p}$ is given by

$$\|F\|_{\mathscr{H}_p} = \begin{cases} \sup_{r\in(0,1)} \left(\frac{1}{2\pi} \int_{\mathbb{R}} |F(re^{i\vartheta})|^p d\vartheta \right)^{\frac{1}{p}} & if \qquad 0 < p < \infty \\ \sup_{z\in\mathbb{D}} |F(z)| & if \qquad p = \infty \end{cases}.$$

Note that the definition of the Hardy space involves all $r \in (0,1)$.

Let $\mathcal{B}$ denote the class of bounded operators. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the trace and the Hilbert-Schmidt class equipped with the norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$ correspondingly.

We define the modified determinant $D(z), z \in \mathbb{D}$ by

$$D(z) = \det\Big[(I + Y_0(\lambda(z)))e^{-Y_0(\lambda(z))}\Big], \qquad z \in \mathbb{D},$$

$$Y_0(\lambda) = V^{\frac{1}{2}}(\Delta - \lambda)^{-1}|V|^{\frac{1}{2}}, \quad V = V^{\frac{1}{2}}|V|^{\frac{1}{2}}, \quad \lambda \in \Lambda = \mathbb{C} \setminus [-d, d],$$

The determinant $D(z), z \in \mathbb{D}$ is well defined, since $Y_0(\lambda) \in \mathcal{B}_2$. It is well known that if $\lambda_0 \in \Lambda$ is an eigenvalue of H, then $z_0 = z(\lambda_0) \in \mathbb{D}$ is a zero of D with the same multiplicity.

We present preliminary results.

Theorem C1. *Let a complex potential V satisfy (1). Then the determinant $D(z),\ z \in \mathbb{D}$ is analytic, Hölder up to the boundary, and satisfies*

$$\|D\|_{\mathscr{H}_\infty} \leqslant e^{C_*^2 \|V\|_p^2/2}, \tag{3}$$

where C_* depends on d, p only. It has $N \leqslant \infty$ zeros $\{z_j\}_{j=1}^N$ in the disc $\mathbb{D}$, such that

$$0 < r_0 = |z_1| \leqslant |z_2| \leqslant |z_3| \leqslant ...., \quad \sum_{j=1}^N (1 - |z_j|) < \infty.$$

Moreover, the function $\psi(z) = \log D(z)$ is analytic in $\mathbb{D}_{r_0}$ and has the following Tailor series (here $a = \frac{2}{d}$)

$$\psi(z) = -\psi_2 z^2 - \psi_3 z^3 - \psi_4 z^4 + ..., \qquad \text{as} \quad |z| < r_0,$$

$$\psi_2 = \frac{a^2}{2} \operatorname{Tr} V^2, \quad \psi_3 = \frac{a^3}{3} \operatorname{Tr}\ V^3, ...$$

This Theorem and below firstly were proved jointly with Ari Laptev for potentials under the condition: $V \in \ell^{\frac{2}{3}}(\mathbb{Z}^d)$. We improve these results using the above joint results with Jacob Möller.

Define the Blaschke product $B(z), z \in \mathbb{D}$ by

$$\begin{aligned} B(z) &= \prod_{j=1}^{N} \frac{|z_j|}{z_j} \frac{(z_j - z)}{(1 - \bar{z}_j z)}, \qquad && \textit{if} \qquad N \geqslant 1, \\ B &= 1, && \textit{if} \qquad N = 0. \end{aligned} \tag{4}$$

We describe the well-known properties of the Blaschke product B in this case. The Blaschke product $B(z)$ converges absolutely for $\{|z| < 1\}$ and satisfies the function $B \in \mathscr{H}_\infty$ with $\|B\|_{\mathscr{H}_\infty} \leqslant 1$. The Blaschke product B has the Taylor series

$$\log B(z) = B_0 - B_1 z - B_2 z^2 - ..., \qquad as \qquad z \to 0,$$

where $B_0 = \log B(0) = \log \prod |z_j| < 0$ and $B_n, n \geqslant 1$ satisfy

$$B_1 = \sum_{1 \leqslant j \leqslant N} \left(\frac{1}{z_j} - \overline{z}_j \right), ..., \qquad B_n = \frac{1}{n} \sum_{1 \leqslant j \leqslant N} \left(\frac{1}{z_j^n} - \overline{z}_j^n \right), ...$$

$$|B_n| \leqslant \frac{2}{r_0^n} \sum_{1 \leqslant j \leqslant N} (1 - |z_j|).$$

where $\sum_{1 \leqslant j \leqslant N} (1 - |z_j|) < \infty$.

We describe the canonical representation of the determinant $D(z)$ and the basic properties of its zeros.

Theorem C2. *Let a complex potential V satisfy (1). Then there exists a singular measure $\nu \geqslant 0$ on $[-\pi, \pi]$ such that the determinant D has a canonical factorization for all $|z| < 1$ given by*

$$\begin{aligned} D(z) &= B(z)e^{-K_\nu(z)}e^{K_D(z)}, \\ K_\nu(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\nu(t), \\ K_D(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|D(e^{it})|dt, \end{aligned} \qquad (5)$$

where the function $\log|D(e^{it})| \in L^1(-\pi, \pi)$. and the measure ν satisfies

$$\operatorname{supp}\nu \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}. \qquad (6)$$

We describe .

Theorem C3. Trace formulas *Let a complex V satisfy (1). Then*

$$0 \leqslant \frac{\nu(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|D(e^{it})|dt, \qquad (7)$$

$$B_n = \psi_n + \frac{1}{\pi}\int_{\mathbb{T}} e^{-int} d\mu(t), \qquad n = 1, 2, ... \qquad (8)$$

where $B_0 = \log B(0) = \log \prod |z_j| < 0$ and B_n, ψ_n are Taylor coefficients of B and $\psi = \log D$, and the measure $d\mu(t) = \log|D(e^{it})|dt - d\nu(t)$. In particular, (recall $\psi_1 = 0$)

$$\sum_{j=1}^{N}\left(\frac{1}{z_j} - \overline{z}_j\right) = \frac{1}{\pi}\int_{\mathbb{T}} e^{-it} d\mu(t),$$

The simple inequality $1 - x \leqslant -\log x$ for $\forall\, x \in (0,1]$ and the identity

$$\frac{\nu(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|D(e^{it})|dt \geqslant 0,$$

where $B_0 = \log B(0) = \log \prod |z_j| < 0$ and the estimate

$$\|D\|_{\mathscr{H}_\infty} \leqslant e^{C_*^2 \|V\|_p^2/2}$$

imply the following estimates.

Corollary C4. *Let a complex V satisfy (1). Then*

$$\frac{\nu(\mathbb{T})}{2\pi} + \sum(1 - |z_j|) \leqslant \frac{C_*^2}{2}\|V\|_p^2.$$

and in particular, if $\operatorname{Im} V \geqslant 0$, then

$$\sum_{j=1}^{N} \operatorname{Im} \lambda_j \leqslant C_*^2 \|V\|_p^2.$$

Here there are a lot of open problems. For example:

1) when the operator H has a finite number of eigenvalues?

2) when the operator H has a infinite number of eigenvalues? (to construct an example). If yes, to describe accumulation points.

3) To describe the a singular measure $\nu \geqslant 0$ on $[-\pi, \pi]$.

Trace formulas for real potentials. We consider Schrödinger operators $H = \Delta + V$ with real potentials V under the condition (1). In this case all eigenvalues λ_j and the numbers $z_j, j = 1, ..., N$ are real. Thus we have the same modified determinant $D(z)$ and D belongs to the Hardy space $\mathscr{H}_\infty$. We formulate results about trace formulas and estimates.

Corollary C5. *Let V be real and satisfy (1). Then*

$$\frac{\nu(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |D(e^{it})| dt \geqslant 0, \tag{9}$$

$$\sum_{j=1}^{N} |\lambda_j^2 - d^2|^{\frac{1}{2}} \operatorname{sign} \lambda_j = \frac{d}{2\pi} \int_{\mathbb{T}} e^{-it} d\mu(t), \tag{10}$$

$$\sum_{j=1}^{N} |\lambda_j^2 - d^2|^{\frac{1}{2}} |\lambda_j| = \frac{1}{2} \operatorname{Tr} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} e^{-2it} d\mu(t), \tag{11}$$

where $B_0 < 0$

Corollary. *Let V be real and satisfy (1). Then*

$$\frac{\nu(\mathbb{T})}{2\pi} - B_0 \leqslant \frac{C_*^2}{2}\|V\|_p^2,,$$

$$|\sum_{j=1}^{N} \varkappa_j| \leqslant C_*^2\|V\|_p^2,$$

$$\sum_{j=1}^{N} |\lambda_j^2 - d^2|^{\frac{1}{2}}|\lambda_j| \leqslant \frac{1}{2}\,\mathrm{Tr}\ V^2 + \frac{d^2}{4\pi} C_*^2\|V\|_p^2,$$

where $B_0 < 0$ and $\varkappa_j = |\lambda_j^2 - d^2|^{\frac{1}{2}}\,\mathrm{sign}\,\lambda_j$.

A Carleson set E is a closed set of measure zero contained in the unit circle $\mathbb{T}$ for which, if the intervals complementary to E have lengths ε_n

$$\sum \varepsilon_n \log \frac{1}{\varepsilon_n} < \infty.$$

This yields that a closed subset of a Carleson set is itself a Carleson set.

Let the algebra $\mathcal{A}_\infty$ be the class of all functions analytic in the open unit disk $\mathbb{D}$ with all derivatives bounded in $\mathbb{D}$.

Novinger [70] showed that every Carleson set is the set of boundary zeros of a function $f \in \mathcal{A}_\infty$, while Beurling [39] proved that the zeros in $\partial\mathbb{D}$ of any function, analytic in $\mathbb{D}$ and continuous in $\mathbb{D}$ which satisfies a Lipschitz condition on $\mathbb{S}$, must form a Carleson set.

Recall that $\operatorname{supp} \nu \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}$. Then $\operatorname{supp} \nu$ is a Carleson set.

Remarks. Introduce the conformal mapping $\varkappa : \Lambda \to \mathbb{K} = \mathbb{C} \setminus [id, -id]$ by

$$\varkappa = \sqrt{\lambda^2 - d^2}, \qquad \varkappa = \lambda - \frac{d^2}{2\lambda} + \frac{O(1)}{\lambda^3} \quad \text{as } |\lambda| \to \infty.$$

Let the eigenvalues $\lambda_j = \lambda_{j1} + i\lambda_{j2}$ and $\varkappa_j = \sqrt{\lambda_j^2 - d^2} = \varkappa_{j1} + i\varkappa_{j2}$. We rewrite the trace formulas in terms of $\lambda_j, \varkappa_j$. In particular,

$$\sum_{j=1}^{N} \Big(\operatorname{Re} \varkappa_j + i \operatorname{Im} \lambda_j \Big) = \frac{d}{2\pi} \int_{\mathbb{T}} e^{-it} d\mu(t),$$

$$\sum_{j=1}^{N} \Big[(\varkappa_{j1}\lambda_{j1} - \lambda_{j2}\varkappa_{j2}) + i(\lambda_{j1}\lambda_{j2} + \varkappa_{j1}\varkappa_{j2}) \Big] = \frac{1}{2} \operatorname{Tr} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} e^{-2it} d\mu(t).$$

Scattering on periodic metric graphs.

We consider the Schrödinger operator $H = H_0 + Q$ on $L^2(\mathcal{G})$, where $H_0 = \Delta_M$ and the potential $Q \in L^2(\mathcal{G}) \cap L^1(\mathcal{G})$ is real. Here $L^1(\mathcal{G})$ is the space of all functions $f = (f_{\mathbf{e}})_{\mathbf{e}\in\mathcal{E}}$ on $\mathcal{G}$ equipped with the norm $\|f\|_{L^1(\mathcal{G})} = \sum_{\mathbf{e}\in\mathcal{E}} \|f_{\mathbf{e}}\|_{L^1(\mathbf{e})}$. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the trace and the Hilbert-Schmidt class equipped with the norm $\|\cdot\|_{\mathbf{B}_1}$ and $\|\cdot\|_{\mathbf{B}_2}$, respectively.

In Theorem **??** we show that $QR_0(k) \in \mathbf{B}_2$ for all $k \in \mathbb{C}_+$. This implies that the operator H is self-adjoint on $\mathscr{D}(H_0)$. For any $k \in \mathbb{C}_+$ we put

$$\begin{aligned} &R_0(k) = (H_0 - k^2)^{-1}, && R(k) = (H - k^2)^{-1}, \\ &Y_0(k) = |Q|^{1/2}\, R_0(k)\, Q^{1/2}, && Q^{1/2} = |Q|^{1/2} \operatorname{sign} Q. \end{aligned} \qquad (12)$$

Below we show that the operator $Y_0(k)$ belongs to the trace class. Thus, we can define the Fredholm determinant D by

$$D(k) = \det(I + Y_0(k)), \qquad k \in \mathbb{C}_+. \qquad (13)$$

Korotyaev+Saburova (Rev. Math. Phys. 2020) prove

Theorem. *Let $Q \in L^2(\mathcal{G}) \cap L^1(\mathcal{G})$ be real. Then $QR_0(k) \in \mathbf{B}_2$ for all $k \in \mathbb{C}_+$ and $H = H_0 + Q$ is self-adjoint on $\mathscr{D}(H_0)$ and*

$$R(k) - R_0(k),\ Y_0(k) \in \mathbf{B}_1 \qquad \forall\, k \in \mathbb{C}_+,$$

and the determinant $D(k) = \det(I + Y_0(k))$ is well-defined and is analytic in $\mathbb{C}_+$ and the limit $D(k + i0)$ exists for almost all $k \in \mathbb{R}$. Furthermore, the wave operators

$$W_\pm = s - \lim\ e^{itH} e^{-itH_0} P_{ac}(H_0) \qquad as \qquad t \to \pm\infty,$$

exist and are complete, i.e., the range of $W_\pm$ is equal to $\mathscr{H}_{ac}(H)$ and $\sigma_{ac}(H) = \sigma_{ac}(H_0)$. Moreover, the S-operator given by

$$S = W_+^* W_-$$

is unitary on $\mathscr{H}_{ac}(H_0)$ and the corresponding S-matrix $S(k)$ for almost all $k^2 \in \sigma_{ac}(H_0)$ satisfies

$$S(k) = I_k - 2\pi i A(k), \quad A(k) \in \mathbf{B}_1, \quad \det S(k) = \frac{\overline{D}(k + i0)}{D(k + i0)}.$$

Korotyaev+Saburova (Rev. Math. Phys. 2020) consider the Schrödinger operator $H = H_0 + V, H_0 = \Delta_M$ on a periodic metric graph G and prove the following: *Let $V \in L^1(G)$. Then the wave operators*

$$W_\pm = s - \lim e^{itH} e^{-itH_0} P_{ac}(H_0) \qquad as \qquad t \to \pm\infty$$

exist and are complete, i.e., $\mathscr{H}_{\mathrm{ac}}(H) = \mathrm{Ran}\, W_\pm$. But here we have not results about $\sigma_{sc}(H)$ and $\sigma_{pp}(H)$. In order to describe these spectrum we consider the metric Laplacian Δ_M on the cubic metric graph

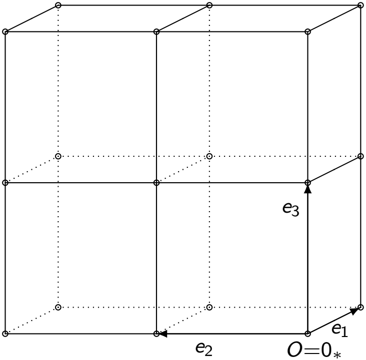
$$\mathbb{L}^d = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \,\middle|\, \exists j \text{ s.t. } x_i \in \mathbb{Z}, \text{ for all } i \neq j\right\}.$$

An edge starting at $n \in \mathbb{Z}^d$ and ending at $n' \in \mathbb{Z}^d$ will be denoted as the ordered pair $(n, n') \in \mathcal{E}$. The set $\mathbb{L}^d$ consists of the elements of $\mathbb{Z}^d$ together with elements from the edges constituting the edge set $\mathcal{E}$ given by

$$\mathcal{E} = \left\{(n, n + e_j) \,\middle|\, n \in \mathbb{Z}^d, j = 1, \ldots, d\right\}, \tag{14}$$

and $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ is the standard orthonormal basis.

(a)



(b)

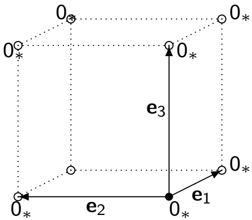


Figure: *a*) Lattice $\mathbb{L}^3$; *b*) the fundamental graph $\mathbb{L}^3_*$.

For each edge $\mathbf{e} \in \mathcal{E}$ we define an orientation. An edge starting at a vertex u and ending at a vertex v from $\mathcal{V}$ will be denoted as the ordered pair $\mathbf{e} = (u, v)$. Each edge $\mathbf{e}$ of G will be identified with the segment $[0, 1]$. This identification introduces a local coordinate $t \in [0, 1]$ along each edge. Thus, we give an orientation on the edge. Note that the spectrum of Laplacians on metric graphs does not depend on the orientation of graph edges. For each function y on G we define a function $y_{\mathbf{e}} = y\big|_{\mathbf{e}}$, $\mathbf{e} \in \mathcal{E}$. We identify each function $y_{\mathbf{e}}$ on $\mathbf{e}$ with a function on $[0, 1]$ by using the local coordinate $t \in [0, 1]$. We define the self-adjoint positive metric Laplacian Δ_M on $L^2(G)$ by

$$(\Delta_M y)_{\mathbf{e}} = -y''_{\mathbf{e}}, \qquad y = (y_{\mathbf{e}})_{\mathbf{e} \in \mathcal{E}}, \quad (y''_{\mathbf{e}})_{\mathbf{e} \in \mathcal{E}} \in L^2(G),$$

where y satisfies the so-called Kirchhoff conditions at each vertex.

It is well known that

$$\sigma(\Delta_M) = \sigma_{ac}(\Delta_M) \cup \sigma_{flat}(\Delta_M), \quad \sigma_{ac}(\Delta_M) = \mathbb{R}_+,$$

$\sigma_{flat}(\Delta_M)$ is the set of all eigenvalues of infinite multiplicity (flat bands). We have $\sigma_{flat}(\Delta_M) = \{(\pi n)^2, n \in \mathbb{N}\}$.

Let V be the operator multiplication in $L^2(\mathbb{L}^d)$. We will write $V = (V(y))_{y \in \mathbb{L}^d}$, where

$$(Vf)(y) = V(y)f(y) \quad \text{for} \quad y \in \mathbb{L}^d.$$

Consider a Schrödinger operator

$$H = \Delta_M + V.$$

We define the **threshold set** $\mathcal{T}$ by
$$\mathcal{T} = \left\{ (\pi n + \arccos(\tfrac{2s-d}{d}))^2 : (n,s) \in \mathbb{N}_0 \times \{0, 1, ..., d\} \right\}.$$
Here $\mathbb{N}_0 = \{0, 1, 2, ...\}$. We describe the spectral properties of H using the Mourre approach.

Theorem. (EK+Möller+Rasmussen) *Let V satisfy*

$$|V(y)| \leqslant C(1+|n|^2)^{-a}, a > 2 \forall\ (y,n) \in \mathbb{L}^d \times \mathbb{Z}^d, \qquad |y-n| \leqslant 1.$$

Then
i) The pure-point spectrum $\sigma_{pp}(H)$ *is of finite multiplicity in* $\mathbb{R}_+ \setminus \mathcal{T}$ *and can only accumulate at* $\mathcal{T}$ *and* $\sigma_{sc}(H) = \emptyset$.
ii) Then the operator-valued function $|V|^{\frac{1}{2}}(\Delta_M - k^2)^{-1}|V|^{\frac{1}{2}}$ *is analytic in the domain* $\mathbb{C}_+$ *and uniformly Hölder up to the boundary in the domain* $\mathbb{C}_+ \setminus \{|k - \pi n| < \varepsilon, n \in \mathbb{N}\}$ *for any* $\varepsilon > 0$.
iii) The wave operators

$$W_\pm = s - \lim e^{itH} e^{-itH_0} \qquad as \qquad t \to \pm\infty$$

exist and are complete, i.e., $\mathscr{H}_{\mathrm{ac}}(H) = \mathrm{Ran}\, W_\pm$.
The proof is based on a Mourre estimate.