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DIRECT AND INVERSE SCATTERING FOR DIFFERENTIAL SYSTEMS WITH SINGULARITY

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Consider the following system of differential equations:

$$y' = (\rho B + x^{-1} A + q(x))y, \; x > 0 \tag{1}$$

with the spectral parameter ρ and $n \times n$ $(n > 2)$ matrices $A, B, q(x), x \in (0, \infty)$, where A, B are constant.

Condition 1. $B = diag(b_1, \dots, b_n)$, the entries $b_1, \dots, b_n$ are nonzero distinct complex numbers such that $\sum_{j=1}^{n} b_j = 0$. Any three of $b_1, \dots, b_n$ are noncollinear.

Condition 2. The matrix A is off-diagonal. Its eigenvalues $\{\mu_j\}_{j=1}^{n}$ are distinct and such that $\mu_j - \mu_k \notin \mathbb{Z}$ for $j \neq k$, moreover, $\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2 < \dots < \mathrm{Re}\mu_n$, $\mathrm{Re}\mu_k \neq 0$.

Condition 3. The matrix function $q(x)$ is off-diagonal, $q_{jk}(\cdot) \in X_p := L_1(0, \infty) \cap L_p(0, \infty)$, $p > 2$.

Unperturbed system:

$$y' = (\rho B + x^{-1} A)y \tag{2}$$

Related scalar differential operator:

$$\ell y := y^{(n)} + \sum_{k=0}^{n-2} \left(q_k(x) + \frac{\nu_k}{x^{n-k}} \right) y^{(k)} \tag{3}$$

Related topics:

- Radial equations arising from PDEs and systems of PDEs having rotational symmetry
- Weighted ODEs of the form $\ell y = \lambda r(x) y$ or $\ell_1 y = \lambda \ell_2 y$ having turning point
- Singular solutions for integrable PDEs, obtained, for instance via Bäcklund transform

Inverse spectral problems for (1) and (3) with $n = 2$ (since 1953): Stashevskaya, Krein, Faddeev, Marchenko, ..., Albeverio, Hryniv, Kostenko, Teschl, Bondarenko, ...

Inverse spectral problems for (1) and (3) with $n > 2$ (since 1992): Yurko, Kudishin, Fedoseev.

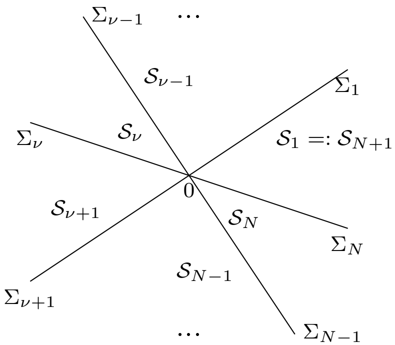
Applicability conditions of the V.A. Yurko approach. The functions $q_{kj}(\cdot)$ are absolutely continuous, integrable on the semi-axis $(0, \infty)$ and such that:

$$\int_0^1 \left| x^{\mu_1 - \mu_n} q_{kj}(x) \right| dx < \infty$$

Let Σ be the following union of lines through the origin in $\mathbb{C}$:

$$\Sigma = \bigcup_{(k,j):j\neq k} \{\rho : \mathrm{Re}(\rho b_j) = \mathrm{Re}(\rho b_k)\}.$$

Then $\mathbb{C} \setminus \Sigma$ can be presented as a union of the sectors $\mathcal{S}_\nu, \nu = \overline{1, N}$.



Consider some arbitrary sector $\mathcal{S}_\nu$. Let $R_1, \dots, R_n$ be the ordering of the numbers $b_1, \dots, b_n$ such that $\mathrm{Re}(\rho R_1) < \mathrm{Re}(\rho R_2) \cdots < \mathrm{Re}(\rho R_n)$ for any $\rho \in \mathcal{S}_\nu$. We denote by $\mathfrak{f}$ the permutation matrix such that $(R_1, \dots, R_n) = (b_1, \dots, b_n)\mathfrak{f}$.

Definition of the Weyl–type solutions

Let k and $\rho \in \mathcal{S}_\nu$ are (arbitrary) fixed. Function $y(x), x \in (0, \infty)$ is called k-th Weyl–type solution if it satisfies (1) and the following asymptotics hold:

$$y(x) = O\left(x^{\mu_k}\right), x \to 0, \; y(x) = \exp(\rho R_k x)(\mathfrak{f}_k + o(1)), x \to \infty.$$

Solutions of the unperturbed system

We start with the unperturbed system for $\rho = 1$:

$$y' - x^{-1}Ay = By \tag{4}$$

and (complex) $x \in \mathcal{S}_\nu$. The following fundamental matrices for system (4) are known to exist:

- $c(x) = (c_1(x), \dots, c_n(x))$, where

  $$c_k(x) = x^{\mu_k}\hat{c}_k(x),$$

  $\det c(x) \equiv 1$ and all $\hat{c}_k(\cdot)$ are entire functions;
- $e(x) = (e_1(x), \dots, e_n(x))$, where

  $$e_k(x) = \mathrm{e}^{xR_k}(\mathfrak{f}_k + x^{-1}\eta_k(x)),\ \ \eta_k(x) = O(1), x \to \infty, x \in \mathcal{S}_\nu.$$

Weyl–type solutions for the unperturbed system

Condition I. For all $\nu = \overline{1, N}$, $k = \overline{1, n}$ the numbers

$$\Delta_k^0 := \det(e_1(x), \ldots, e_{k-1}(x), c_k(x), \ldots, c_n(x))$$

are not equal to 0.

Under Condition I unperturbed system (4) has the (unique) fundamental matrix $\psi_0(x)$, $x \in \mathcal{S}_\nu$ such that

$$\psi_{0,k}(tx) = \mathrm{e}^{txR_k}(\mathfrak{f}_k + o(1)), t \to \infty, x \in \mathcal{S}_\nu, \ \psi_{0,k}(x) = O(x^{\mu_k}), x \to 0.$$

For unperturbed system (2) with $\rho \in \mathcal{S}_\nu$, $x > 0$ we introduce the following fundamental matrices:

- $C(x, \rho) := c(\rho x)$;
- $E(x, \rho) := e(\rho x)$;
- $\Psi_0(x, \rho) := \psi_0(\rho x)$.

Basic idea for general case (R. Beals, P. Deift, C. Tomei, 1988)

Suppose the Weyl–type solutions $\{\Psi_k(x,\rho)\}_{k=1}^n$ are already constructed. Then:

- for any multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ the tensor-valued function $\Psi_\alpha(x,\rho) := \Psi_{\alpha_1}(x,\rho) \wedge \dots \wedge \Psi_{\alpha_m}(x,\rho)$ satisfies some auxiliary system of ODEs;
- the tensors Ψ_1, $\Psi_1 \wedge \Psi_2$, $\Psi_1 \wedge \Psi_2 \wedge \Psi_3$, ... have a minimal growth as $x \to \infty$;
- the tensors Ψ_n, $\Psi_{n-1} \wedge \Psi_n$, $\Psi_{n-2} \wedge \Psi_{n-1} \wedge \Psi_n$, ... have a minimal growth as $x \to 0$.

Notations

For given $n \times n$ matrix M $M^{(m)}$ denote an operator acting in $\wedge^m \mathbb{C}^n$ so that for any vectors $u_1, \dots, u_m$ the following identity holds:

$$M^{(m)}(u_1 \wedge u_2 \wedge \cdots \wedge u_m) = \sum_{j=1}^{m} u_1 \wedge u_2 \wedge \cdots \wedge u_{j-1} \wedge M u_j \wedge u_{j+1} \wedge \cdots \wedge u_m.$$

Denote by $\mathcal{A}_m$ the set of all ordered multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, $\alpha_j \in \{1, 2, \dots, n\}$. For a set of vectors $u_1, \dots, u_n$ from $\mathbb{C}^n$ and a multi-index $\alpha \in \mathcal{A}_m$ we define

$$u_\alpha := u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m}.$$

Let $a_1, \dots, a_n$ be a numerical sequence. For $\alpha \in \mathcal{A}_m$ we define

$$a_\alpha := \sum_{j \in \alpha} a_j, \;\; a^\alpha := \prod_{j \in \alpha} a_j.$$

Notations

For $k \in \overline{1,n}$ we denote

$$\overrightarrow{a}_k := \sum_{j=1}^{k} a_j,\ \overleftarrow{a}_k := \sum_{j=k}^{n} a_j,\ \overrightarrow{a}^k := \prod_{j=1}^{k} a_j,\ \overleftarrow{a}^k := \prod_{j=k}^{n} a_j.$$

For a multi-index α the symbol α' denotes the ordered multi-index that complements α to $(1, 2, \dots, n)$. We note that Assumption 1 implies, in particular, that $\sum_{k=1}^{n} \mu_k = \sum_{k=1}^{n} R_k = 0$ and therefore for any multi-index α one has $R_{\alpha'} = -R_\alpha$ and $\mu_{\alpha'} = -\mu_\alpha$. For $h \in \wedge^n \mathbb{C}^n$ we define $|h|$ as a constant in the following representation:

$$h = |h| \mathfrak{e}_1 \wedge \mathfrak{e}_2 \wedge \cdots \wedge \mathfrak{e}_n.$$

Notations

Introduce the function:

$$W_0(\xi) := (1-|\xi|)\xi + |\xi|^2,\ |\xi| \le 1,\ W_0(\xi) := \left(W_0\left(\xi^{-1}\right)\right)^{-1},\ |\xi| > 1,$$

$$W_k(\xi) := W_0\left(\xi^{\mu_k}\right)\exp(R_k\xi),\ |\xi| \le 1, \quad W_k(\xi) := \exp(R_k\xi),\ |\xi| > 1,$$

$k = \overline{1, n}$.
We denote by $W(\xi)$ the following diagonal matrix:

$$W(\xi) := diag(W_1(\xi), \ldots, W_n(\xi))$$

Fundamental tensors

We consider the following Volterra integral equations:

$$Y(x) = T_k^0(x,\rho) + \int\limits_0^x G_{n-k+1}(x,t,\rho)\left(q^{(n-k+1)}(t)Y(t)\right)dt, \tag{4}$$

$$Y(x) = F_k^0(x,\rho) - \int\limits_x^\infty G_k(x,t,\rho)\left(q^{(k)}(t)Y(t)\right)dt, \tag{5}$$

where

$$T_k^0(x,\rho) := C_k(x,\rho)\wedge\cdots\wedge C_n(x,\rho),$$

$$F_k^0(x,\rho) := E_1(x,\rho)\wedge\cdots\wedge E_k(x,\rho) = \Psi_{0,1}(x,\rho)\wedge\cdots\wedge\Psi_{0,k}(x,\rho)$$

and $G_m(x,t,\rho)$ denote an operator acting in $\wedge^m\mathbb{C}^n$ as follows:

$$G_m(x,t,\rho)f = \sum_{\alpha\in\mathcal{A}_m}\chi_\alpha\left|f\wedge\Psi_{0,\alpha'}(t,\rho)\right|\Psi_{0,\alpha}(x,\rho), \quad \chi_\alpha = |\mathfrak{f}_\alpha\wedge\mathfrak{f}_{\alpha'}|$$

Fundamental tensors

Theorem 1. For any $\rho \in \mathcal{S}_\nu$ equations (4), (5) have the unique solutions $T_k(q,x,\rho)$ and $F_k(q,x,\rho)$ respectively. The following representations hold:

$$T_k(q,x,\rho) = T_k^0(x,\rho) + \overleftarrow{W}^k(\rho x)\hat{T}_k(q,x,\rho),$$

$$F_k(q,x,\rho) = F_k^0(x,\rho) + \overrightarrow{W}^k(\rho x)\hat{F}_k(q,x,\rho),$$

$\hat{T}_k, \hat{F}_k \in C(\mathcal{X}_p, BC([0,\infty), C_0(\overline{\mathcal{S}}_\nu)))$ and for any ray $l = \{\rho = zt, t \in [0,\infty)\}$, $z \in \overline{\mathcal{S}}_\nu \setminus \{0\}$ the restrictions $\hat{T}_k\Big|_l, \hat{F}_k\Big|_l \in C(\mathcal{X}_p, BC([0,\infty), \mathcal{H}(l)))$.
Here the symbol $\mathcal{X}_p$ denotes the Banach space of all off-diagonal matrices with entries from X_p, $\mathcal{H}(l) = C_0(l) \cap L_2(l)$.

Weyl–type solutions

Define $\Delta_k(q,\rho) := |F_{k-1}(q,x,\rho) \wedge T_k(q,x,\rho)|$.

Theorem 2. For any $\rho \in \mathcal{S}_\nu$ such that $\Delta_k(\rho) \neq 0$ there exists and is unique the Weyl–type solution $\Psi_k(x,\rho)$. For each fixed $x > 0$ Ψ_k is a unique vector with the properties:

$$F_{k-1} \wedge \Psi_k = F_k, \;\; \Psi_k \wedge T_k = 0.$$

Theorem 3. The following relation holds:

$$(\Psi_0(x,\rho))^{-1}\, \Psi(q,x,\rho) = W^{-1}(\rho x)\beta(q,x,\rho)W(\rho x),$$

where

$$\beta_{jk}(q,x,\rho) = \frac{\delta_{jk} + d_{jk}(q,x,\rho)}{1 + d_k(q,x,\rho)},$$

$d_{jk}, d_k \in C(\mathcal{X}_p, BC([0,\infty), C_0(\overline{\mathcal{S}}_\nu)))$ and for any ray $l = \{\rho = zt, t \in [0,\infty)\}$, $z \in \overline{\mathcal{S}}_\nu \setminus \{0\}$ the restrictions $d_{jk}\big|_l$, $d_k|_l$ belong to $C(\mathcal{X}_p, BC([0,\infty), H(l)))$.

For a set $L \subset \mathbb{C}$ we define $G_0^p(L)$ as the class of functions $q(\cdot) \in \mathcal{X}_p$ such that $\prod_{k=1}^{n} \Delta_k(\rho) \neq 0$ for all $\rho \in L$.

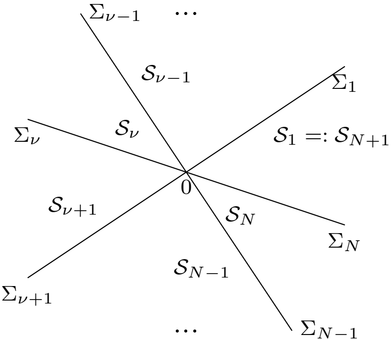
Theorem 4. *Suppose $p > 2$. Then the following representation holds:*

$$\Psi_k(q, x, \rho) = \Psi_{0k}(x, \rho) + W_k(\rho x)\hat{\Psi}_k(q, x, \rho),$$

where $\hat{\Psi}_k \in C(G_0^p(l), C([0, T], \mathcal{H}(l)))$ for any ray $l = \{\rho = zt, t \in [0, \infty)\}$, $z \in \overline{\mathcal{S}_\nu} \setminus \{0\}$ and any segment $[0, T], 0 < T < \infty$.

Scattering data

For a function $f = f(\rho)$, $\rho \in \mathbb{C} \setminus \Sigma$ and $\rho_0 \in \Sigma'$ the expressions $f^{\pm}(\rho_0)$ will denote the limits (if exist): $f^{\pm}(\rho_0) = \lim\limits_{\varepsilon \to +0} f(\rho_0 \pm i\rho_0\varepsilon)$. In some cases, for the sake of brevity, we write $f(\rho_0)$ instead of $f^{-}(\rho_0)$.



Suppose $q(\cdot) \in L_1(0,\infty)$, $\Psi = \Psi(x,\rho) := (\Psi_k(q,x,\rho))_{k=1}^{n}$, $x \in (0,\infty)$, $\rho \in \mathbb{C} \setminus \Sigma$. If $\rho \in \Sigma_\nu$ is such that $\Delta_k(\rho) \neq 0$, $\Delta_k^{+}(\rho) \neq 0$, $k = \overline{1,n}$ then all the limits $\Psi^{\pm}(x,\rho)$ exist and there exists the matrix $v = v(\rho)$ such that:

$$\Psi^{+}(x,\rho) = \Psi^{-}(x,\rho)v(\rho) \tag{5}$$

Properties of scattering data

For $\rho \in \Sigma'$ define $\Pi(\rho)$ as the permutation matrix such that

$$(R_1^+(\rho), \dots, R_n^+(\rho)) = (R_1^-(\rho), \dots, R_n^-(\rho))\Pi(\rho),$$

or, equivalently, $R^+(\rho) = \Pi^{-1}(\rho)R^-(\rho)\Pi(\rho)$. Clear that $\Pi(\rho)$ is block diagonal and constant on each ray Σ_ν. In what follows we call a matrix function $\Pi(\rho)$, $\rho \in \Sigma'$ Π-*diagonal* if it is block diagonal and its block structure is the same as the block structure of $\Pi(\rho)$.
Define $I_- := \{k : \mathrm{Re}(\rho R_k) = \mathrm{Re}(\rho R_{k+1})\}$ (note that for $k \in I_-$ one has: $\mathrm{Re}(\rho R_{k-1}) < \mathrm{Re}(\rho R_k)$ if $k > 1$).

Theorem 5. *For each $\rho \in \Sigma'$ $v(\rho)$ is low triangular and Π-diagonal. Moreover, for each $k \in I_-$ one has:*

1. $v_{k+1,k}(\rho) \equiv 1$;
2. $v_{kk}(\rho)v_{k+1,k+1}(\rho) \equiv -1$.

Theorem 6. *Suppose that for all $\nu = \overline{1, N}$ the following condition is satisfied:*

$$\lim_{\rho \to 0, \rho \in \overline{\mathcal{S}}_\nu} \prod_{k=1}^{n} \Delta_k(\rho) \neq 0.$$

Then for all $\nu = \overline{1, N}$ there exist the limits $\lim\limits_{\rho \to 0, \rho \in \Sigma_\nu} v(\rho)$, which do not depend on the potential $q(\cdot)$.

Properties of scattering data

Definition *We say that* $q(\cdot) \in \mathcal{X}_p$ *belongs to* $G_0^p(\Sigma)$ *if*

$$\prod_{k=1}^{n} \Delta_k^{\pm}(q, \rho) \neq 0$$

for all $\rho \in \Sigma$.

Definition $\mathcal{H}(\Sigma)$ is the space of functions $\varphi \in L_2(\Sigma)$ such that:

1. for any $\nu = \overline{1, N}$ the restriction $\varphi(\rho)|_{\rho \in \Sigma_\nu}$ is continuous;
2. there exists the limits $\lim\limits_{\rho \to 0, \rho \in \Sigma_\nu} \varphi(\rho)$, $\lim\limits_{\rho \to \infty, \rho \in \Sigma_\nu} \varphi(\rho)$;
3. $\lim\limits_{\rho \to \infty, \rho \in \Sigma_\nu} \varphi(\rho) = 0$.

For $\varphi \in \mathcal{H}(\Sigma)$ we set $\|\varphi\| := \|\varphi\|_{L_2(\Sigma)} + \|\varphi\|_{C_0(\Sigma)}$.
Define $\mathcal{H}_0(\Sigma) := \{\varphi(\cdot) \in \mathcal{H}(\Sigma) : \lim\limits_{\rho \to 0, \rho \in \Sigma_\nu} \varphi(\rho) = 0, \nu = \overline{1, N}$.

Theorem 7. *Suppose* $p > 2$. *Then* $v(\cdot, \cdot) - v_0(\cdot) \in C(G_0^p(\Sigma), \mathcal{H}_0(\Sigma))$. *Here and below* $v_0(\rho) := v(0, \rho)$.

Formulation of Inverse Scattering Problem

Definition *We say that* $q(\cdot) \in \mathcal{X}_p$ *belongs to* G_0^p *if for any* $\nu = \overline{1, N}$ $q(\cdot) \in G_0^p\left(\overline{\mathcal{S}}_\nu\right)$.

Problem 1. *Given* $v(q, \rho)$, $\rho \in \Sigma'$ *for some* $q(\cdot) \in G_0^p$, *recover* $q(\cdot)$.

In what follows we call the matrix function $v(q, \rho)$ the *scattering data*.

Solution of Problem 1. Main equation.

The Cauchy operators:

$$Cf(\rho) := \frac{1}{2\pi i}\int\limits_{\Sigma}\frac{d\zeta}{\zeta-\rho}f(\zeta),\ \ \rho\in\mathbb{C}\setminus\Sigma;$$

$$C^{\pm}f(\rho) := (Cf)^{\pm}(\rho).$$

Fix an arbitrary $q(\cdot)\in G_0^p$ and set $P(x,\rho)=P(q,x,\rho)$, where $P(q,x,\rho) := \Psi(q,x,\rho)\left(\Psi_0(x,\rho)\right)^{-1}$ (*spectral mapping matrix*).

Theorem 8. *Define* $\hat{P}(x,\rho) := P^{+}(x,\rho)-P^{-}(x,\rho)$. *For each fixed* $x>0$ *the following assertions are true:*

1. $\hat{P}(x,\cdot)$ *is the solution of the equation* $\mathbf{A}(x)\varphi=\hat{V}(x,\cdot)$ *unique in* $L_2(\Sigma)$, *where* $\hat{V}(x,\rho) := V(x,\rho)-I$, $V(x,\rho) := \Psi_0(x,\rho)v(\rho)v_0^{-1}(\rho)\Psi_0^{-1}(x,\rho)$ *and linear operator* $\mathbf{A}(x)$ *act in* $L_2(\Sigma)$ *by the formula:*

$$\mathbf{A}(x)\varphi(\rho) := \left(C^{+}\varphi\right)(\rho)-\left(C^{-}\varphi\right)(\rho)V(x,\rho);$$

2. *the operator* $\mathbf{A}(x)$ *is invertible.*

Solution of Problem 1. Reconstruction formula for smooth potentials.

Theorem 9. *Let* $q(\cdot) \in G_0^p$ *be absolutely continuous function compactly supported in* $(0, \infty)$*. Then the following reconstruction formula is true:*

$$q(x) = \frac{1}{2\pi i} \int\limits_{\Sigma} \left[B, \hat{P}(x,\rho)\right] d\rho,$$

where the integral in the right hand side is considered as the following limit:

$$\frac{1}{2\pi i} \int\limits_{\Sigma} \left[B, \hat{P}(x,\rho)\right] d\rho := \lim_{r \to \infty} \frac{1}{2\pi i} \int\limits_{\Sigma \cap \{|\rho| < r\}} \left[B, \hat{P}(x,\rho)\right] d\rho.$$

Here and below $[\cdot,\cdot]$ denotes the matrix commutator: $[M_1, M_2] := M_1 M_2 - M_2 M_1$.

Solution of Problem 1. Reconstruction formula.

By $L_2^+(\Sigma)$ we denote the space Π-upper triangular matrix functions with the elements from $L_2(\Sigma)$, $\mathcal{H}_0^+(\Sigma) := \mathcal{H}_0(\Sigma) \cap L_2^+(\Sigma)$. By $\mathcal{H}_0^\Pi(\Sigma)$ we denote the space of low triangular Π - diagonal matrix functions with the elements from $\mathcal{H}_0(\Sigma)$.

Lemma 1. *Define the bilinear operator:*

$$\mathbf{\Phi}(u,\varphi)(x) := \frac{1}{2\pi i}\left[B, \int_\Sigma d\rho \left(C^-\varphi(x,\cdot)\right)(\rho)\hat{V}(u,x,\rho)\right],$$

where

$$\hat{V}(u,x,\rho) := \Psi_0(x,\rho)u(\rho)\Psi_0^{-1}(x,\rho).$$

Then:

1. $\mathbf{\Phi} : \mathcal{H}_0^\Pi(\Sigma) \times C([0,\infty), L_2(\Sigma)) \to C[0,\infty)$ *is continuous;*
2. *for any* $u \in \mathcal{H}_0^\Pi(\Sigma)$, $\varphi \in C([0,\infty), L_2(\Sigma))$ *as* $r \to \infty$ $\mathbf{\Phi}_r(u,\varphi) \to \mathbf{\Phi}(u,\varphi)$, *where:*

$$\mathbf{\Phi}_r(u)\varphi(x) := \frac{1}{2\pi i}\int_\Sigma d\rho\, \theta^-(|\rho|-r)\left[B, \left(C^-\varphi(x,\cdot)\right)(\rho)\hat{V}(u,x,\rho)\right].$$

Solution of Problem 1. Reconstruction formula.

Lemma 2. *Define the family of linear operators:*

$$\mathbf{F}_r f(x) := \frac{1}{2\pi i} \int\limits_{\Sigma} d\rho\, \theta^-(|\rho| - r) \left[B, \Psi_0(x,\rho) f(\rho) \Psi_0^{-1}(x,\rho)\right],$$

$r > 0$. *Then:*

1. *for each* $r > 0$ $\mathbf{F}_r \in \mathcal{L}\left(\mathcal{H}_0^{\Pi}(\Sigma), L_{2,loc}(0,\infty]\right)$;
2. *there exists a strong limit*

$$\mathbf{F} = s - \lim_{r\to\infty} \mathbf{F}_r \in \mathcal{L}\left(\mathcal{H}_0^{\Pi}(\Sigma), L_{2,loc}(0,\infty]\right).$$

Theorem 10. *Let* $v(\rho) = v(q,\rho)$ *be the scattering data for some* $q(\cdot) \in G_0^p$, $\hat{v}(\rho) = v(\rho) v_0^{-1}(\rho) - I$, $\hat{P}(x,\rho) = P^+(q,x,\rho) - P^-(q,x,\rho)$. *Then the reconstruction formula holds:*

$$q = \mathbf{\Phi}(\hat{v}, \hat{P}) + \mathbf{F}\hat{v}.$$

Characterization of scattering data

We say that a matrix function $v = v(\rho)$, $\rho \in \Sigma$ belongs to the class $\mathbf{V}$ if:

1. $v(\cdot) - v_0(\cdot) \in \mathcal{H}_0^{\Pi}(\Sigma)$;
2. each nontrivial diagonal block of $v(\rho)$ located in the rows with indices k and $k+1$, where $k \in I_-$, and has the form:

$$\begin{pmatrix} v_{kk} & 0 \\ 1 & v_{k+1,k+1} \end{pmatrix},$$

   where $v_{kk} v_{k+1,k+1} = -1$.

Note that if $v(\cdot)$ is the scattering data for some $q(\cdot) \in G_0^p$ then $v(\cdot) \in \mathbf{V}$.

For two functions $v_1(\cdot) \in \mathbf{V}$ and $v_1(\cdot) \in \mathbf{V}$ we set $dist(v_1, v_2) := \|v_1 - v_2\|_{\mathcal{H}_0^{\Pi}(\Sigma)}$.

Characterization of scattering data

For a matrix function $v = v(\rho)$, $\rho \in \Sigma$ define:

$$\hat{v}(\rho) := v(\rho)v_0^{-1}(\rho) - I,$$

$$V = V(v, x, \rho) := \Psi_0(x, \rho)v(\rho)v_0^{-1}(\rho)(\Psi_0(x, \rho))^{-1},$$

$$\hat{V}(v, x, \rho) := V(v, x, \rho) - I = \Psi_0(x, \rho)\hat{v}(\rho)(\Psi_0(x, \rho))^{-1}.$$

For arbitrary $v \in \mathbf{V}$, $x \in [0, \infty)$ define the operators:

$$\mathbf{A}(v, x)f(\rho) := C^+ f(\rho) - (C^- f)(\rho)V(v, x, \rho) = f(\rho) - (C^- f)(\rho)\hat{V}(v, x, \rho).$$

Define also:

$$\mathbf{p}(v, x, \cdot) := (\mathbf{A}(v, x))^{-1}\hat{V}(v, x, \cdot),$$

$$\mathbf{q}(v, \cdot) := \mathbf{\Phi}(\hat{v}(\cdot), \mathbf{p}(v, \cdot, \cdot)) + \mathbf{F}\hat{v}(\cdot).$$

Characterization of scattering data. Main theorem.

Theorem 11. *For given* $v(\cdot) \in \mathbf{V}$ *to be the scattering data for some* $q(\cdot) \in G_0^p$ *it is necessary and sufficient that:*

1. *for each fixed* $x \in [0, \infty)$ *the operator* $\mathbf{A}(v, x)$ *is invertible;*
2. *for each* $k = \overline{1, n}$ *there exists a function* $\delta_k(\rho)$, $\rho \in \mathbb{C} \setminus \Sigma$, *which is analytic in* $\mathbb{C} \setminus \Sigma$ *and such that:*
   - *for each* $\nu = \overline{1, N}$ *the function* $\rho^{\mu_k} \delta_k(\rho)$ *admits a continuous extension onto* $\overline{\mathcal{S}}_\nu$*;*
   - $\rho^{\mu_k} \delta_k(\rho) \neq 0$ *for all* $\rho \in \overline{\mathcal{S}}_\nu$, $\nu = \overline{1, N}$*;*
   - *for* $\rho \in \Sigma'$ *the following conjugation condition satisfied:* $\delta^-(\rho) = v_{kk}(\rho) \delta_k^+(\rho)$*;*
   - *for* $\rho \to \infty$, $\rho \in \mathbb{C} \setminus \Sigma$ *the asymptotics* $\delta(\rho) = \delta_0(\rho)(I + o(1))$ *holds, moreover,* $\delta^\pm(\cdot)(\delta_0^\pm(\cdot))^{-1} - I \in L_2(\Sigma)$. *Here* $\delta_0(\rho)$ *is the diagonal matrix such that* $\Psi_0(x, \rho)\delta_0(\rho) = (\mathfrak{h} + o(1)) x^\mu$ *as* $x \to 0$, $\mathfrak{h} = (\mathfrak{h}_1, \ldots, \mathfrak{h}_n)$.
3. $\mathbf{q}(v, \cdot) \in \mathcal{X}_p$.

Sufficient conditions

Denote $\mathbf{V}_{00} := (v_0 + C^\infty_{00}(\Sigma)) \cap \mathbf{V}$.

The following theorem shows that in the case $v(\cdot) \in \mathbf{V}_{00}$ condition 3 is satisfied if condition 1 is satisfied.

Theorem 12. *Let $v(\cdot) \in \mathbf{V}_{00}$ is such that condition 1 of the theorem above is satisfied. Then $\mathbf{q}(v, x)$ is continuous in $x \in [0, \infty)$ and the estimate $\mathbf{q}(v, x) = O(x^{-m})$ as $x \to \infty$, is holds with arbitrary $m \geq 0$.*

The theorem above allows to obtain *sufficient conditions* for the Problem 1 solvability.

Corollary *There exists $\varepsilon_0 > 0$ such that any $v(\cdot) \in \mathbf{V}_{00}$: $\|v - v_0\|_{L_\infty(\Sigma)} < \varepsilon_0$ is the scattering data for some $q(\cdot) \in G^p_0$.*