The Limiting Absorption Principle and Continuity Properties of the Spectral Shift Function for Massless Dirac-Type Operators

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Analysis and Mathematical Physics (AMP) 2024

August 5–17, 2024, online

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In Memoriam: Dima Yafaev (1948–2024)

Motivation and Topics Discussed:

Motivation:

- **Index Theory for Non-Fredholm Operators, Witten Index.**
- **Spectral Theory for Schrödinger and Dirac-type Operators in $\mathbb{R}^n$, $n \geqslant 2$.**
- **Scattering Theory, Spectral Shift Function.**

Topics Discussed:

- **Threshold Behavior (i.e., Zero-Energy Resonances and Eigenvalues).**
- **A Limiting Absorption Principle.**
- **Absolutely Continuous Spectrum.**
- **Spectral Shift Function (SSF) and its Continuity Properties.**

A bit of Motivation: The Fredholm Index

Definition (Fredholm operators).

Let T be a closed and densely defined operator in $\mathcal{H}$. Then T is **Fredholm** if $\operatorname{ran}(T)$ is closed in $\mathcal{H}$ and $\dim(\ker(T)) + \dim(\ker(T^*)) < \infty$.

If T is **Fredholm**, its **index** (denoted by $\operatorname{ind}(T)$), is defined as

$$\begin{aligned} \operatorname{ind}(T) &= \dim(\ker(T)) - \dim(\ker(T^*)) \\ &= \dim(\ker(T^*T)) - \dim(\ker(TT^*)). \end{aligned}$$

Some Facts. Suppose T is a closed and densely defined operator in $\mathcal{H}$. Then,

(i) T is **Fredholm** if and only if T^* is and $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

(ii) T is **Fredholm** if and only if there exists $\varepsilon > 0$ such that $\inf(\sigma_{ess}(T^*T)) \geqslant \varepsilon$ **and** $\inf(\sigma_{ess}(TT^*)) \geqslant \varepsilon$. (Note. The **"and"** is crucial here!)

A bit of Motivation: The Fredholm Index

Some Facts (contd.). Suppose T is a closed and densely defined operator in $\mathcal{H}$. Then,

(iii) T **Fredholm**, S relatively compact w.r.t. T (e.g., $S(T - z_0 I_{\mathcal{H}})^{-1}$ compact in $\mathcal{H}$ for some $z_0 \in \rho(T)$), then $T + S$ is **Fredholm** and $\text{ind}(T + S) = \text{ind}(T)$.

$\longrightarrow$ **Stability** of the **Fredholm index** w.r.t. **additive relatively compact perturbations**. Think, **"topological invariance"** .... one of the exciting properties of the Fredholm index! Roughly speaking, **local changes** in the coefficients of a Fredholm PDE operator **will not change its index.**

(iv) S and T **Fredholm**, such that ST is densely defined, then ST is **Fredholm** and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.

More Motivation: The Witten Index and SSF

Perhaps, the best way to formally introduce the real-valued **Krein–Lifshitz spectral shift function (SSF)** on $\mathbb{R}$, $\xi(\,\cdot\,; H, H_0)$, for a pair of self-adjoint operators (H, H_0) in $\mathcal{H}$, is to show what it can do: It computes **traces** for "appropriate" functions f as follows:

$$\boxed{\operatorname{tr}_{\mathcal{H}}(f(H) - f(H_0)) = \int_{\mathbb{R}} f'(\lambda)\, \xi(\lambda; H, H_0)\, d\lambda,}$$

E.g., if H_0, H are bounded from below by some $c\, I_{\mathcal{H}}$ and $\left[e^{-tH} - e^{-tH_0}\right]$ **is trace class** in $\mathcal{H}$ for some $t_0 > 0$, then

$$\operatorname{tr}_{\mathcal{H}}\left(e^{-tH} - e^{-tH_0}\right) = -t \int_{[c,\infty)} e^{-t\lambda} \xi(\lambda; H, H_0)\, d\lambda, \quad t > 0.$$

Similarly, if **resolvent** differences are **trace class.**

Definition of $\xi(\lambda; H, H_0)$: E.g., if $\operatorname{dom}(H) = \operatorname{dom}(H_0)$ and $(H - H_0)(H_0 - z_0 I_{\mathcal{H}})^{-1}$ **is trace class** in $\mathcal{H}$. Then, for a.e. $\lambda \in \mathbb{R}$,

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(\ln\big(\det_{\mathcal{H}}\big(I_{\mathcal{H}} + (H - H_0)(H_0 - (\lambda + i\varepsilon) I_{\mathcal{H}})^{-1}\big)\big)\big).$$

This works well for ODEs, but for PDEs we need to use heavier machinery.

Motivation: The Witten Index and SSF (contd.)

Here $\det_{\mathcal{H}}(I_{\mathcal{H}} + T)$ denotes the **Fredholm determinant** of a **trace class operator** T in $\mathcal{H}$.

Next, let T be a closed, linear, densely defined operator in $\mathcal{H}$ and suppose that for some $t_0 > 0$ (and hence for **all** $t > t_0$), $\left[e^{-t_0 T^* T} - e^{-t_0 T T^*}\right]$ is trace class. Then the **(semigroup regularized) Witten index** $W_s(T)$ of T is defined by

$$W_s(T) = \lim_{t \to \infty} \operatorname{tr}_{\mathcal{H}} \left(e^{-t T^* T} - e^{-t T T^*}\right),$$

whenever this limit exists. This represents an interesting **extension** of the notion of the **Fredholm index** to some classes of **non-Fredholm operators**.

If $\xi(\,\cdot\,; T T^*, T^* T)$ is continuous from above at $\lambda = 0$, then $W_s(T)$ exists and $W_s(T) = \xi(0_+; T T^*, T^* T)$.

Normalization: If H_0, H are bounded from below by some $c\, I_{\mathcal{H}}$, then one typically **normalizes** $\xi(\,\cdot\,; H, H_0)$ such that

$$\xi(\lambda; H, H_0) = 0, \quad \lambda < c.$$

A more sophisticated approach shows that it is sufficient that 0 is a **right** and **left Lebesgue point** of $\xi(\,\cdot\,; T T^*, T^* T)$.

Motivation: The Witten Index and SSF (contd.)

Theorem A (Consistency in the case of Fredholm operators).

Assume that for some $t_0 > 0$, $\left[e^{-t_0 T^* T} - e^{-t_0 TT^*}\right]$ **is trace class** and suppose that T is **Fredholm**. Then $W_s(T)$ exists and

$$\operatorname{ind}(T) = W_s(T) = \xi(0_+; TT^*, T^*T).$$

Remark. Generally, the **Witten index**, $W_s(T)$ is not integer-valued. E.g., in a concrete 2d magnetic field system $W_s(T)$ has the meaning of **magnetic flux** $F \in \mathbb{R}$, an arbitrary real number! **Still, one can prove a stability result:**

F.G., B. Simon, *Topological invariance of the Witten index*, J. Funct. Anal. **79**, 91–102 (1988),

showed that $W_s(T)$ has **stability properties** w.r.t. **additive perturbations** similar to the Fredholm index, replacing the **relative compactness** assumption on the perturbation by **"appropriate" relative trace class** conditions.

$\longrightarrow$ **Stability** of the **Witten index** w.r.t. **additive relatively trace class perturbations**. Think again, **"topological invariance"** .... i.e., **local** changes of coefficients in PDE operators will not affect the **Witten index!**

Motivation: The Witten Index and SSF (contd.)

The actual Witten index we're interested in is still a bit removed from this setup. However, we can reduce it's value to that of $[\xi(0_+; H, H_0) + \xi(0_-; H, H_0)]/2$, where H_0, H are appropriate free and interacting massless Dirac operators:

$$H_0 = \alpha \cdot (-i\nabla), \quad H = H_0 + V$$

in $[L^2(\mathbb{R}^n)]^N$, $n \in \mathbb{N}$, $n \geqslant 2$, with α, V being $N \times N$-matrices, $N = 2^{\lfloor (n+1)/2 \rfloor}$, and the matrix-valued potential V suitably decaying at infinity (more details soon). The principal aim then is to express the (resolvent or semigroup) regularized Witten index of the following non-Fredholm operator $\boldsymbol{D_A}$ in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ (a model operator studied by **Atiyah–Patodi–Singer, Robbin–Salamon,** etc., given by (with $t \in \mathbb{R}$)

$$\boldsymbol{D_A} = \frac{d}{dt} + \boldsymbol{A}, \quad \mathrm{dom}(\boldsymbol{D_A}) = W^{1,2}\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big) \cap \mathrm{dom}(\boldsymbol{A_-}),$$

in terms of the spectral shift function $\xi(\lambda; H, H_0)$ at $\lambda = 0_\pm$, where

$$\boldsymbol{A} = \boldsymbol{A_-} + \boldsymbol{B} \equiv \int_{\mathbb{R}}^{\oplus} dt\, A_- + \int_{\mathbb{R}}^{\oplus} dt\, B(t), \quad \mathrm{dom}(\boldsymbol{A}) = \mathrm{dom}(\boldsymbol{A_-}).$$

Motivation: The Witten Index and SSF (contd.)

Here $\boldsymbol{A}$, $\boldsymbol{A}_-$, $\boldsymbol{A}_+$, $\boldsymbol{B}$, and $\boldsymbol{B}_+$ in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$ are generated with the help of the Dirac-type operators H, H_0 and potential matrices V as **direct integral operators** as follows:

$$A(t) = A_- + B(t),\ t \in \mathbb{R}, \quad A_- \equiv H_0, \quad A_+ = A_- + B_+ \equiv H,$$
$$B(t) = b(t) B_+,\ t \in \mathbb{R}, \quad B_+ = V,$$

in $[L^2(\mathbb{R}^n)]^N$, assuming $b(\,\cdot\,)$ is a smooth step function satisfying

$$b^{(k)} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dt),\ k \in \mathbb{N}_0, \quad b' \in L^1(\mathbb{R}; dt),$$
$$\lim_{t \to \infty} b(t) = 1, \quad \lim_{t \to -\infty} b(t) = 0.$$

In particular, $A_\pm$ are the asymptotes of the family $A(t)$, $t \in \mathbb{R}$, as $t \to \pm\infty$ in the norm resolvent sense.

In this context,

$$L^2(\mathbb{R}; \mathcal{H}) = \int_{\mathbb{R}}^{\oplus} dt\, \mathcal{H} \quad \text{and} \quad \boldsymbol{T} = \int_{\mathbb{R}}^{\oplus} dt\, T(t)$$

represent direct integrals of Hilbert spaces and operators.

Motivation: The Witten Index and SSF (contd.)

Introducing the nonnegative, self-adjoint operators (the analog of T^*T and TT^*)

$$\boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}}, \quad \boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^*$$

in $L^2\big(\mathbb{R}; [L^2(\mathbb{R}^n)]^N\big)$, the principal aims is to express the (semigroup) regularized Witten index $W_s(\boldsymbol{D}_{\boldsymbol{A}})$ of $\boldsymbol{D}_{\boldsymbol{A}}$ in terms of spectral shift functions and prove the formula

$$W_s(\boldsymbol{D}_{\boldsymbol{A}}) = \xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = [\xi(0_+; H, H_0) + \xi(0_-; H, H_0)]/2.$$

Here the notation $\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1)$ indicates that 0 is a **right Lebesgue point** for $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$.

This explains our interest in $\xi(\lambda; H, H_0)$, $\lambda \in \mathbb{R}$.

Next we discuss the framework that eventually permits the realization of these ideas to the concrete case of the **multi-dimensional, massless** Dirac-type operators H, H_0:

(Massless) Dirac-Type Operators: Background

Hypotheses. Let $n \in \mathbb{N}$, $n \geqslant 2$.

(i) Set $N = 2^{\lfloor (n+1)/2 \rfloor}$ and let α_j, $1 \leqslant j \leqslant n$, $\alpha_{n+1} := \beta$, denote $n+1$ anti-commuting Hermitian (**Clifford**) $N \times N$ matrices with squares equal to I_N, that is,

$$\alpha_j^* = \alpha_j, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k} I_N, \quad 1 \leqslant j,k \leqslant n+1, \quad I_N \text{ the unit matrix in } \mathbb{C}^N.$$

(ii) Introduce in $[L^2(\mathbb{R}^n)]^N$ the **free massless** Dirac operator (with $\partial_j = \partial/\partial x_j$)

$$H_0 = \alpha \cdot (-i\nabla) = \sum_{j=1}^n \alpha_j(-i\partial_j), \quad \mathrm{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N.$$

(iii) Next, consider the self-adjoint matrix-valued potential $V = \{V_{\ell,\ell'}\}_{1 \leqslant \ell,\ell' \leqslant N}$ satisfying for some fixed $\rho \in (1,\infty)$, $C \in (0,\infty)$,

$$V \in [L^\infty(\mathbb{R}^n)]^{N \times N}, \quad |V_{\ell,\ell'}(x)| \leqslant C[1+|x|]^{-\rho} \text{ for a.e. } x \in \mathbb{R}^n,\ 1 \leqslant \ell,\ell' \leqslant N.$$

Under these assumptions on V, the **interacting massless** Dirac operator H in $[L^2(\mathbb{R}^n)]^N$ is defined via

$$H = H_0 + V, \quad \mathrm{dom}(H) = \mathrm{dom}(H_0) = [W^{1,2}(\mathbb{R}^n)]^N.$$

Note. These hypotheses are general enough to permit **electromagnetic fields**!

(Massless) Dirac-Type Operators: Some Properties

Notation: $[L^2(\mathbb{R}^n)]^N = L^2(\mathbb{R}^n;\mathbb{C}^N)$, $[W^{1,2}(\mathbb{R}^n)]^N = W^{1,2}(\mathbb{R}^n;\mathbb{C}^N)$, etc.
Then H_0 and H are self-adjoint in $[L^2(\mathbb{R}^n)]^N$, with **essential spectrum** given by

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma(H_0) = \mathbb{R},$$

since V is **relatively compact** w.r.t. H_0. In addition,

$$\sigma_{ac}(H_0) = \mathbb{R}, \quad \sigma_p(H_0) = \sigma_{sc}(H_0) = \emptyset.$$

Compare with the **massive** free Dirac operator in $[L^2(\mathbb{R}^n)]^N$, with $\beta = \alpha_{n+1}$,

$$H_0(m) = H_0 + m\,\beta, \quad \mathrm{dom}(H_0(m)) = [W^{1,2}(\mathbb{R}^n)]^N, \ \text{ mass } \ m > 0,$$

and the corresponding **interacting massive** Dirac operator in $[L^2(\mathbb{R}^n)]^N$ given by

$$H(m) = H_0(m) + V = H_0 + m\,\beta + V, \quad \mathrm{dom}(H(m)) = [W^{1,2}(\mathbb{R}^n)]^N, \ m > 0,$$
$$\sigma_{ess}(H(m)) = \sigma_{ess}(H_0(m)) = \sigma(H_0(m)) = (-\infty, -m] \cup [m, \infty), \quad m > 0,$$
$$\sigma_{ac}(H_0(m)) = (-\infty, -m] \cup [m, \infty), \quad \sigma_p(H_0(m)) = \sigma_{sc}(H_0(m)) = \emptyset, \quad m > 0.$$

One observes the identity

$$H_0(m)^2 = I_N\big[-\Delta + m^2 I_{[L^2(\mathbb{R}^n)]^N}\big], \quad \mathrm{dom}\left(H_0(m)^2\right) = [W^{2,2}(\mathbb{R}^n)]^N, \quad m \geqslant 0.$$

(Massless) Dirac-Type Operators: Green's Matrix

From now on we put $\boxed{m = 0}$ and almost exclusively focus on the **massless** case.

The **Green's function (matrix)** of H_0: Assuming $z \in \mathbb{C}_+$, $x, y \in \mathbb{R}^n$, $x \neq y$, $n \in \mathbb{N}$, $n \geqslant 2$, then

$$\begin{aligned} G_0(z; x, y) &:= (H_0 - zI)^{-1}(x, y) \\ &= i4^{-1}(2\pi)^{(2-n)/2}|x-y|^{2-n} z\,[z|x-y|]^{(n-2)/2} H^{(1)}_{(n-2)/2}(z|x-y|) I_N \\ &\quad - 4^{-1}(2\pi)^{(2-n)/2}|x-y|^{1-n}[z|x-y|]^{n/2} H^{(1)}_{n/2}(z|x-y|)\, \alpha \cdot \frac{(x-y)}{|x-y|}. \end{aligned}$$

$H^{(1)}_\nu(\,\cdot\,)$ the **Hankel function** of the first kind with index $\nu \geqslant 0$.

$G_0(z; \,\cdot\,, \,\cdot\,)$ of H_0 continuously extends to $z \in \overline{\mathbb{C}_+}$. In addition, the limit $z \to 0$ exists

$$\lim_{\substack{z \to 0, \\ z \in \overline{\mathbb{C}_+} \setminus \{0\}}} G_0(z; x, y) = i2^{-1}\pi^{-n/2}\Gamma(n/2)\, \alpha \cdot \frac{(x-y)}{|x-y|^n}, \quad x, y \in \mathbb{R}^n, \, x \neq y,$$

and (in contrast to $m > 0$ if $n = 2$!) no blow up occurs for all $n \in \mathbb{N}$, $n \geqslant 2$.

Spectral Behavior of H:

Since

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma(H_0) = \sigma_{ac}(H_0) = \mathbb{R}, \qquad \sigma_p(H_0) = \sigma_{sc}(H_0) = \emptyset,$$

one asks, **"What could possibly go wrong with $\sigma(H)$?"** Surely, also $\sigma_{ac}(H) = \mathbb{R}$, right? **Well, not so fast!**

Even though, $\sigma_{ess}(H) = \mathbb{R}$, one could still have **embedded eigenvalues**, perhaps, even existence of some **singular continuous spectrum**, $\sigma_{sc}(H)$, a much dreaded possibility within some circles of Mathematical Physicists, though, very welcome by others! (The feeling is community dependent, e.g., atomic scattering theory vs. condensed matter physics .....)

In fact, zero eigenvalues at energy $z = 0$ have explicitly been constructed, so one cannot take anything for granted!

Eventually, we will show that

$$\sigma_{ac}(H) = \mathbb{R}, \quad \sigma_{sc}(H) = \emptyset, \text{ and } \sigma_p(H) \cap \mathbb{R}\backslash\{0\} = \emptyset,$$

but it takes a rather long path getting there.

Threshold Behavior of H:

Note. $\sigma_{ess}(H(m)) = \sigma_{ess}(H_0(m)) = (-\infty, -m] \cup [m, \infty)$, $m > 0$. Thus, the **mass gap**, $[-m, m]$ closes as $m \downarrow 0$, hence,

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = (-\infty, 0] \cup [0, \infty) = \mathbb{R},$$

but, 0 **is still a distinguished and rather "delicate" point!** Hence, We'll discuss **zero-energy (= threshold) resonances** and **eigenvalues** next:

Hypotheses. Let $n \in \mathbb{N}$, $n \geqslant 2$.

Assume the a.e. self-adjoint matrix-valued potential $V = \{V_{\ell,\ell'}\}_{1 \leqslant \ell,\ell' \leqslant N}$ satisfies for some $C \in (0, \infty)$,

$$V \in [L^\infty(\mathbb{R}^n)]^{N \times N},$$

$$|V_{\ell,\ell'}(x)| \leqslant C(1 + |x|)^{-2} \text{ for a.e. } x \in \mathbb{R}^n, \ 1 \leqslant \ell, \ell' \leqslant N.$$

In addition, alluding to the **polar decomposition** of $V(\,\cdot\,)$ (i.e., $V(\,\cdot\,) = U_V(\,\cdot\,)|V(\,\cdot\,)|$) in the following symmetrized form, we suppose that

$$V = V_1^* V_2 = |V|^{1/2} U_V |V|^{1/2}, \text{ where } V_1 = V_1^* = |V|^{1/2}, \quad V_2 = U_V |V|^{1/2}.$$

(This is a typical **quadratic form** assumption.)
We continue with the **threshold behavior**, that is, the $z = 0$ behavior, of H:

Threshold Behavior of H (contd.):

Definition.

(i) The point 0 is called a **zero-energy eigenvalue** of H if $H\Psi = 0$ has a distributional solution Ψ satisfying $\Psi \in \operatorname{dom}(H) = [W^{1,2}(\mathbb{R}^n)]^N$ (equivalently, $\ker(H) \supsetneqq \{0\}$).

(ii) The point 0 is called a **zero-energy (or threshold) resonance** of H if

$$\ker\left(\left[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\right]\right) \supsetneqq \{0\},$$

(a **Birman–Schwinger-type** operator!) and if there exists $0 \neq \Phi \in \ker\left(\left[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*}\right]\right)$ such that Ψ defined by

$$\begin{aligned}\Psi(x) &= -\left((H_0 - (0+i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1}V_1^*\Phi\right)(x)\\ &= -i2^{-1}\pi^{-n/2}\Gamma(n/2)\int_{\mathbb{R}^n} d^n y\,|x-y|^{-n}[\alpha(x-y)]V_1(y)^*\Phi(y)\end{aligned}$$

(for a.e. $x \in \mathbb{R}^n$, $n \geqslant 2$) is a distributional solution of $H\Psi = 0$ s.t. $\Psi \notin [L^2(\mathbb{R}^n)]^N$.

(iii) 0 is called a **regular point** for H if it is **neither a zero-energy eigenvalue nor a zero-energy resonance** of H. (The **generic** case.)

Threshold Behavior of H (contd.):

While the point 0 being regular for H is the generic situation, **zero-energy eigenvalues** and/or **resonances** are **exceptional** cases.

Theorem B (F.G., R. Nichols, 2020).

(i) If $n = 2$, there are precisely four possible cases:

Case (I): 0 is **regular** for H.

Case (II): 0 is a (possibly at most twice degenerate) **resonance** of H. In this case, the resonance functions Ψ satisfy

$$\Psi \in [L^q(\mathbb{R}^2)]^2, \quad q \in (2,\infty) \cup \{\infty\}, \quad \nabla\Psi \in [L^2(\mathbb{R}^2)]^{2\times 2},$$
$$\Psi \notin [L^2(\mathbb{R}^2)]^2.$$

Case (III): 0 is a (possibly degenerate) **eigenvalue** of H. In this case, the corresponding eigenfunctions $\Psi \in \operatorname{dom}(H) = \big[W^{1,2}(\mathbb{R}^2)\big]^2$ of $H\Psi = 0$ also satisfy

$$\Psi \in [L^q(\mathbb{R}^2)]^2, \quad q \in [2,\infty) \cup \{\infty\}.$$

Case (IV): A possible **mixture** of Cases (II) and (III).

Threshold Behavior of H (contd.):

Theorem B contd. (F.G., R. Nichols, 2020).

(ii) If $n \in \mathbb{N}$, $n \geqslant 3$, there are precisely two possible cases:

Case (I): 0 is **regular** for H.

Case (II): 0 is a (possibly degenerate) **eigenvalue** of H. In this case, the corresponding eigenfunctions $\Psi \in \operatorname{dom}(H) = \left[W^{1,2}(\mathbb{R}^n)\right]^N$ of $H\Psi = 0$ also satisfy

$$\Psi \in \left[L^q(\mathbb{R}^n)\right]^N, \quad q \in \begin{cases} (3/2, \infty) \cup \{\infty\}, & n = 3, \\ (4/3, 4), & n = 4, \\ (2n/(n+2), 2n/(n-2)), & n \geqslant 5. \end{cases}$$

In particular, there are **no** zero-energy resonances of H in dimension $n \geqslant 3$.

(iii) The point 0 is **regular** for H if and only if

$$\ker\left(\left[I_{[L^2(\mathbb{R}^n)]^N} + \overline{V_2(H_0 - (0 + i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}\right]\right) = \{0\}.$$

Note. For **massive** Dirac operators $H(m)$, there are **no** threshold resonances at energies $\pm m$ in dimensions $n \geqslant 5$; also for Schrödinger operators there are **no** zero-energy resonances in dimension $n \geqslant 5$. (But, threshold eigenvalues can exist.)

Threshold Behavior of H (contd.):

A few elements that enter our proofs:

(1) L^p-properties of **Riesz potentials**:

Theorem (E.g., Stein, Singular Integrals and Diff. Prop. of Fcts., 1970).

Let $n \in \mathbb{N}$, $\alpha \in (0, n)$, and introduce the **Riesz potential operator** $\mathcal{R}_{\alpha,n}$ as follows (**fractional Laplacian** on $\mathbb{R}^n$):

$$(\mathcal{R}_{\alpha,n}f)(x) = \big((-\Delta)^{-\alpha/2}f\big)(x) = \gamma(\alpha,n)^{-1}\int_{\mathbb{R}^n} d^n y\, |x-y|^{\alpha-n} f(y),$$

$$\gamma(\alpha,n) = \pi^{n/2}2^{\alpha}\Gamma(\alpha/2)/\Gamma((n-\alpha)/2),$$

for appropriate functions f (see below).

(i) Let $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^n)$. Then the integral $(\mathcal{R}_{\alpha,n}f)(x)$ converges for a.e. $x \in \mathbb{R}^n$.

(ii) Let $1 < p < q < \infty$, $q^{-1} = p^{-1} - \alpha n^{-1}$, and $f \in L^p(\mathbb{R}^n)$. Then there exists $C_{p,q,\alpha,n} \in (0,\infty)$ such that

$$\|\mathcal{R}_{\alpha,n}f\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,q,\alpha,n}\|f\|_{L^p(\mathbb{R}^n)}.$$

Threshold Behavior of H (contd.):

(2) The **Riesz composition formula** (e.g., **Du Plessis**, An Introduction to Potential Theory, 1970),

$$\int_{\mathbb{R}^n} d^n y\, |x-y|^{\alpha-n} |y-w|^{\beta-n} = [\gamma(\alpha,n)\gamma(\beta,n)/\gamma(\alpha+\beta,n)] |x-w|^{\alpha+\beta-n},$$
$$0<\alpha<n,\ 0<\beta<n,\ 0<\alpha+\beta<n,\ x,w\in\mathbb{R}^n,$$

where again $\gamma(\alpha,n) = \pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)/\Gamma((n-\alpha)/2)$.

(3) An estimate taken from a 2010 paper by **Erdogan and Green**:

Lemma.

Let $n \in \mathbb{N}$ and $x_1, x_2 \in \mathbb{R}^n$. If $k, \ell \in [0,n)$, $\varepsilon, \beta \in (0,\infty)$, with $k+\ell+\beta \geqslant n$, and $k+\ell \neq n$, then

$$\int_{\mathbb{R}^n} d^n y\, |x_1-y|^{-k} [1+|y|]^{-(\beta+\varepsilon)} |y-x_2|^{-\ell}$$
$$\leqslant C_{n,k,\ell,\beta,\varepsilon}(x_1,x_2) \cdot \begin{cases} |x_1-x_2|^{-\max\{0,k+\ell-n\}}, & |x_1-x_2| \leqslant 1, \\ |x_1-x_2|^{-\min\{k,\ell,k+\ell+\beta-n\}}, & |x_1-x_2| \geqslant 1, \end{cases}$$

where $C_{n,k,\ell,\beta,\varepsilon}(x_1,x_2) \in (0,\infty)$.

A Limiting Absorption Principle for H:

Using **(strongly) locally Kato-smooth operators**, the following was proved in **A. Carey, F.G., G. Levitina, R. Nichols, F. Sukochev, and D. Zanin, 2021:**

Theorem C (Carey et al., 2021).

Under the previous hypotheses the following hold:

$$\sigma_{ess}(H) = \sigma_{ac}(H) = \mathbb{R},$$
$$\sigma_{sc}(H) = \emptyset, \quad \sigma_s(H) \cap (\mathbb{R}\backslash\{0\}) = \sigma_p(H) \cap (\mathbb{R}\backslash\{0\}),$$

with the **only** possible accumulation points of $\sigma_p(H)$ being 0 and $\pm\infty$. Define

$$\mathcal{N}_\pm = \Big\{\lambda \in \mathbb{R}\backslash\{0\} \,\Big|\, \text{there exists } 0 \neq f \in [L^2(\mathbb{R}^n)]^N \text{ s.t.}$$
$$-f = \overline{V_2(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*} f\Big\},$$

then

$$\mathcal{N}_+ = \mathcal{N}_- := \mathcal{N}_0 = \sigma_p(H) \cap (\mathbb{R}\backslash\{0\}) = \sigma_d(H) \cap (\mathbb{R}\backslash\{0\}),$$

and the (geometric) multiplicities of the eigenvalue $\lambda_0 \in \mathbb{R}\backslash\{0\}$ of H and the eigenvalue -1 of $\overline{V_2(H_0 - (\lambda_0 \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}$ coincide and are finite.

A Limiting Absorption Principle for H (contd.):

Theorem C contd. (Carey et al., 2021).

The operators $\overline{V_1(H_0 - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}$, $\overline{V_1(H - (\lambda \pm i0)I_{[L^2(\mathbb{R}^n)]^N})^{-1} V_1^*}$ are Hölder continuous in norm with respect to λ varying in compact subintervals of $\mathbb{R}\backslash\{0\}$ (resp., $\mathbb{R}\backslash(\{0\} \cup \mathcal{N}_0)$).

Finally, the global **wave operators**

$$W_\pm(H, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0},$$

exist and are complete, that is,

$$\ker(W_\pm(H, H_0)) = \{0\}, \quad \operatorname{ran}(W_\pm(H, H_0)) = E_{H,ac}\mathcal{H},$$

with $E_{H,ac}$ the projection onto the absolutely continuous subspace of H.

This is based on the notion of **(strongly) locally Kato-smooth operators**, an abstract machinery one can find, e.g., in two AMS volumes (1992 and 2010) by **D. Yafaev**. The details are somewhat intense and in the interest of time we take this for granted.

Absence of Embedded Eigenvalues of H:

Assume now in addition that:

(a) For some $R > 0$, $V \in \left[C^1(E_R)\right]^{N\times N}$, where $E_R = \{x \in \mathbb{R}^n \,|\, |x| \geqslant R\}$, and that $(x \cdot \nabla V_{\ell,\ell'})(x) \underset{|x|\to\infty}{=} o(1), \quad 1 \leqslant \ell, \ell' \leqslant N,$ uniformly with respect to directions.

(b) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} |x| \|V(x)\|_{\mathcal{B}(\mathbb{C}^N)} \leqslant C$ for some $C \in (0, (n-1)/2)$ (**smallness** of V !), with $\|\cdot\|_{\mathcal{B}(\mathbb{C}^N)}$ denoting the operator norm of an $N \times N$ matrix in $\mathbb{C}^N$.

Theorem D (H. Kalf, T. Okaji, O. Yamada, 2015).

(i) Assume that V satisfies the additional conditions in (a). Then

$$\sigma_p(H) \subseteq \{0\}.$$

(ii) Assume that V satisfies the additional conditions (a), (b). Then

$$\sigma_p(H) = \emptyset.$$

Note. Combining Thm. D (ii) with our Thm. C, H and H_0 are **unitarily equivalent** via the **wave operators** $W_\pm(H, H_0)$.

SSF in a nutshell: Mark Krein 1953–1962:

Notation. $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$, denote the ℓ^p-based **trace ideals**, i.e., **compact**, linear operators T in $\mathcal{H}$ whose **singular values** $\big($the eigenvalues of $(T^* T)^{1/2}\big)$ are $\ell^p(\mathbb{N})$-summable. ($\mathcal{B}_1(\mathcal{H})$ the **trace class**, $\mathcal{B}_2(\mathcal{H})$ the **Hilbert–Schmidt class**, ...)

Theorem (M. Krein).

Assume

$$\big[(B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \rho(A) \cap \rho(B). \qquad (*)$$

Then there exists $\xi(\,\cdot\,; B, A) \in L^1_{loc}(\mathbb{R}; d\lambda)$ such that $\int_{\mathbb{R}} |\xi(\lambda; B, A)|(1 + \lambda^2)^{-1}\, d\lambda < \infty$ and

$$\operatorname{tr}_{\mathcal{H}}\big((B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}\big) = -\int_{\mathbb{R}} \frac{\xi(\lambda; B, A)\, d\lambda}{(\lambda - z)^2}, \quad z \in \rho(A) \cap \rho(B).$$

The function $\xi(\,\cdot\,; B, A)$ is **unique up to a real constant**.

- Trace formula for $\varphi(\lambda) = (\lambda - z)^{-1}$ and $\varphi(\lambda) = (\lambda - z)^{-k}$.
- Large class of φ's are discussed in **V. Peller '85** (he employs Besov spaces).

SSF in a nutshell: Mark Krein 1953–1962 (contd.)

Corollary. (Eigenvalue counting in essential spectral gaps.)

If $\delta = (a, b)$ and $\overline{\delta} \cap \sigma_{ess}(A) = \emptyset$ then

$$\xi(b_-; B, A) - \xi(a_+; B, A) = \dim(\operatorname{ran}(E_B(\delta))) - \dim(\operatorname{ran}(E_A(\delta))).$$

The Birman–Krein formula.

Assume

$$\big[(B - zI_{\mathcal{H}})^{-1} - (A - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \rho(A) \cap \rho(B). \qquad (*)$$

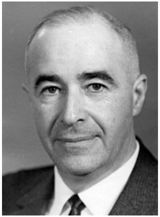
Then the scattering matrix $\{S(\lambda; B, A)\}_{\lambda \in \sigma_{ac}(A)}$ for the pair (B, A) satisfies

$$\det(S(\lambda; B, A)) = e^{-2\pi i \xi(\lambda; B, A)} \quad \text{for a.e. } \lambda \in \sigma_{ac}(A).$$

The Krein–Lifshitz spectral shift function ξ:

"On the shoulders of giants":

Ilya Mikhailovich Lifshitz (January 13, 1917 – October 23, 1982):



Well-known Theoretical Physicist: Worked in solid state physics, electron theory of metals, disordered systems, Lifshitz tails, Lifshitz singularity, the theory of polymers; **introduced the concept of the spectral shift function for finite-rank perturbations in 1952.**

Mark Grigorievich Krein (April 3, 1907 – October 17, 1989):

Mathematician Extraordinaire: One of the giants of 20th century mathematics, Wolf Prize in Mathematics in 1982; **introduced the theory of the spectral shift function in the period of 1953–1963.**

SSF: Generalizations

Theorem (D. R. Yafaev '05).

Assume that for some **odd**, $r \in \mathbb{N}$,

$$\big[(B - zI_{\mathcal{H}})^{-r} - (A - zI_{\mathcal{H}})^{-r}\big] \in \mathcal{B}_1(\mathcal{H}).$$

Then there exists $\xi(\,\cdot\,; B, A) \in L^1_{loc}(\mathbb{R}; d\lambda)$ such that

$\int_{\mathbb{R}} |\xi(\lambda; B, A)|(1 + |\lambda|)^{-(r+1)}\, d\lambda < \infty$ and

$$\operatorname{tr}_{\mathcal{H}}\big((B - zI_{\mathcal{H}})^{-r} - (A - zI_{\mathcal{H}})^{-r}\big) = \int_{\mathbb{R}} \frac{-r}{(\lambda - z)^{r+1}}\, \xi(\lambda; B, A)\, d\lambda,$$
$$z \in \rho(A) \cap \rho(B).$$

Note. (i) **Yafaev** assumes **no** spectral gaps of A $\longrightarrow$ applicable to **massless Dirac-type** operators H, H_0, the prime examples of **non-Fredholm operators**.

(ii) The case $r = 1$ works for ODE operators only. To treat PDE operators, $r = r(n) > 1$ has to be appropriately chosen depending on the space dimension n involved, in fact, the choice $r = n$ works for the massless Dirac operators H, H_0.

SSF: Generalizations (contd.):

To prove our principal result on the pair of **Dirac operators** (H, H_0) we employ the following representation for **spectral shift functions** $\xi(\,\cdot\,; B, A)$ in terms of **modified** (or **regularized**) **Fredholm determinants** which is applicable to the multi-dimensional case:

Hypothesis.

Let A and B be self-adjoint operators in $\mathcal{H}$ with $(B - A) \in \mathcal{B}(\mathcal{H})$.

(i) If $r \in \mathbb{N}$ is odd, assume

$$\big[(B - zI_{\mathcal{H}})^{-r} - (A - zI_{\mathcal{H}})^{-r}\big] \in \mathcal{B}_1(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

and

$$(B - A)(A - zI_{\mathcal{H}})^{-j} \in \mathcal{B}_{(r+1)/j}(\mathcal{H}), \quad j \in \mathbb{N},\ 1 \leqslant j \leqslant r + 1.$$

(ii) If $r \in \mathbb{N}$ is even, assume in addition that for some $0 < \varepsilon < 1/2$,

$$(B - A)\big(A^2 + I_{\mathcal{H}}\big)^{-(r/2)-\varepsilon} \in \mathcal{B}_1(\mathcal{H}).$$

SSF: Generalizations (contd.):

Introduce

$$F_{B,A}(z) := \ln\big(\det_{\mathcal{H},r+1}\big((B - zI_{\mathcal{H}})(A - zI_{\mathcal{H}})^{-1}\big)\big), \quad z \in \mathbb{C}\backslash\mathbb{R},$$

where $\det_{\mathcal{H},r+1}(I_{\mathcal{H}} + T)$ denotes the **modified** (or **regularized**) **Fredholm determinant** for operators T in the **trace ideal** $\mathcal{B}_{r+1}(\mathcal{H})$.

In addition, introduce the analytic function $G_{B,A}(\,\cdot\,)$ in $\mathbb{C}\backslash\mathbb{R}$ such that

$$\frac{d^r}{dz^r} G_{B,A}(z) = \operatorname{tr}_{\mathcal{H}} \left(\frac{d^{r-1}}{dz^{r-1}} \sum_{j=0}^{r-1} (-1)^{r-j} (A - zI_{\mathcal{H}})^{-1} B(z)^{r-j} \right), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Here we used the abbreviation $B(z) = (B - A)(A - zI_{\mathcal{H}})^{-1}$.

Then there exist polynomials $P_{\pm,r-1}$ of degree less than or equal to $r - 1$ such that

$$F_{B,A}(z) = (z - i)^r \int_{\mathbb{R}} \frac{\xi(\lambda; B, A)\, d\lambda}{(\lambda - i)^r} \frac{1}{\lambda - z} + G_{B,A}(z) + P_{\pm,r-1}(z), \quad z \in \mathbb{C}_{\pm}.$$

SSF: Generalizations (contd.):

These preparations finally lead to the following representation for $\xi(\,\cdot\,; B, A)$:

Theorem E (A. Carey et al., 2021).

If $F_{B,A}$ and $G_{B,A}$ have normal boundary values on $\mathbb{R}$, then for a.e. $\lambda \in \mathbb{R}$,

$$\xi(\lambda; B, A) = \pi^{-1}\mathrm{Im}(F_{B,A}(\lambda + i0)) - \pi^{-1}\mathrm{Im}(G_{B,A}(\lambda + i0)) + P_{r-1}(\lambda) \quad \text{for a.e. } \lambda \in \mathbb{R},$$

where P_{r-1} is a polynomial of degree less than or equal to $r - 1$.

Thus, to analyze continuity properties of $\xi(\lambda; B, A)$, one focuses on properties of the **(normal, or nontangential) boundary values** of $F_{B,A}(\lambda + i0)$ and $G_{B,A}(\lambda + i0)$, $\lambda \in \mathbb{R}$.

SSF for the pair of Dirac operators (H, H_0):

Hypotheses. Let $n \in \mathbb{N}$ and suppose that $V = \{V_{\ell,\ell'}\}_{1 \leqslant \ell,\ell' \leqslant N}$ satisfies for some constants $C \in (0, \infty)$ and $\varepsilon > 0$,

$$V \in [L^\infty(\mathbb{R}^n)]^{N \times N}, \quad |V_{\ell,\ell'}(x)| \leqslant C(1+|x|)^{-n-1-\varepsilon} \text{ for a.e. } x \in \mathbb{R}^n,\ 1 \leqslant \ell, \ell' \leqslant N.$$

Note. We now assume much more decay of $V(\,\cdot\,)$ at infinity to guarantee existence of $\xi(\,\cdot\,; H, H_0)$.

In addition, assume that $V(x) = \{V_{\ell,\ell'}(x)\}_{1 \leqslant \ell,\ell' \leqslant N}$ is self-adjoint for a.e. $x \in \mathbb{R}^n$. In accordance with the factorization based on the polar decomposition of V we suppose that $V = V_1^* V_2 = |V|^{1/2} U_V |V|^{1/2}$, where $V_1 = V_1^* = |V|^{1/2}$, $V_2 = U_V |V|^{1/2}$.

Finally, assume that V satisfies condition (a) in Theorem D, i.e.,

(a) For some $R > 0$, $V \in \big[C^1(E_R)\big]^{N \times N}$, where $E_R = \{x \in \mathbb{R}^n \,|\, |x| \geqslant R\}$, and that $(x \cdot \nabla V_{\ell,\ell'})(x) \underset{|x| \to \infty}{=} o(1), \quad 1 \leqslant \ell, \ell' \leqslant N,$ uniformly with respect to directions.

Note. This is our final and complete list of hypotheses on V.

SSF for the pair of Dirac operators (H, H_0) (contd.):

The principal result of **A. Carey, F.G., G. Levitina, R. Nichols, F. Sukochev, and D. Zanin, 2021** then reads as follows:

Theorem F (Carey et al., 2021).

Assume the above hypotheses on V. Then

$$\xi(\,\cdot\,; H, H_0) \in C((-\infty, 0) \cup (0, \infty)),$$

and the left and right limit at zero,

$$\xi(0_\pm; H, H_0) = \lim_{\varepsilon \downarrow 0} \xi(\pm\varepsilon; H, H_0), \ \text{ exists.}$$

In particular, if 0 is a regular point for H, then $\xi(\,\cdot\,; H, H_0) \in C(\mathbb{R})$.

Thus, **Witten Index** applications now are possible in n dimensions, $n \in \mathbb{N}$.

The proof relies on a barrage of **trace norm estimates** of resolvent differences to guarantee the existence of $\xi(\,\cdot\,; H, H_0)$, and, as indicated, on a representation of $\xi(\,\cdot\,; H, H_0)$ in terms of **nontangential boundary values** to the real axis of an underlying **modified (regularized) Fredholm determinant**. It's a long story ......

Based on:

• **A. Carey, F.G., J. Kaad, G. Levitina, R. Nichols, D. Potapov, and F. Sukochev**, *On the Global Limiting Absorption Principle for Massless Dirac Operators*, Ann. H. Poincaré **19**, 1993–2019 (2018).

• **F.G. and R. Nichols**, *On Absence of Threshold Resonances for Schrödinger and Dirac Operators*, Discrete Cont. Dyn. Syst, Ser. S, **13**, 3427–3460 (2020).

• **A. Carey, F.G., G. Levitina, R. Nichols, F. Sukochev, and D. Zanin**, *The Limiting Absorption Principle for Massless Dirac Operators, Properties of Spectral Shift Functions, and an Application to the Witten Index of Non-Fredholm Operators*, Memoirs of the EMS **4** (2023), 213pp.

Thank you!