The direct and inverse scattering problems for the first-order linear discrete system associated with the derivative NLS system

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Direct and inverse scattering problems for the first-order discrete system associated with the derivative NLS system

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Abstract
The direct and inverse scattering problems are analyzed for a first-order discrete system associated with the semi-discrete version of the derivative nonlinear Schrödinger (NLS) system. The Jost solutions, the scattering coefficients, the bound-state dependency and norming constants are investigated and related to the corresponding quantities for two particular discrete linear systems associated with the semi-discrete version of the NLS system. The bound-state data set with any multiplicities is described in an elegant manner in terms of a pair of constant matrix triplets. Several methods are presented to solve the inverse problem to recover the potential values in the first-order discrete system. One of these methods uses a newly derived, standard discrete Marchenko system using as input the scattering data directly coming from the first-order discrete system. This new Marchenko method is presented in a way that it is generalizable to other first-order systems both in the discrete and continuous cases for which a Marchenko system and a Marchenko theory are not yet available. Finally, using the time-evolved scattering data set, the inverse scattering transform is applied on the corresponding semi-discrete derivative NLS system, and in

Outline

1. DDNLS: semi-discrete derivative NLS (nonlinear Schrödinger) system
2. Derivation of DDNLS system by using AKNS method
3. Linear system associated with DDNLS system
4. Contrast with DNLS, continuous case
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6. Marchenko method to solve DDNLS system
7. Contrast with Marchenko method for solve DNLS
8. Explicit solution formulas for DDNLS system using a pair of matrix triplets

DDNLS system

$$\begin{cases} i\dot{q}_n + \dfrac{q_{n+1}}{1 - q_{n+1} r_{n+1}} - \dfrac{q_n}{1 - q_n r_n} - \dfrac{q_n}{1 + q_n r_{n+1}} + \dfrac{q_{n-1}}{1 + q_{n-1} r_n} = 0, \\ i\dot{r}_n - \dfrac{r_{n+1}}{1 + q_n r_{n+1}} + \dfrac{r_n}{1 + q_{n-1} r_n} + \dfrac{r_n}{1 - q_n r_n} - \dfrac{r_{n-1}}{1 - q_{n-1} r_{n-1}} = 0. \end{cases}$$

- $q_n := q_n(t), \quad r_n := r_n(t)$
- $1 - q_n r_n \neq 0, \quad 1 + q_n r_{n+1} \neq 0, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}$
- overdot $\quad$ *t*-derivative

Derivation of DDNLS system by using AKNS method

- $(\mathcal{X}_n, \mathcal{T}_n)$ AKNS pair

$$\mathcal{X}_n = \begin{bmatrix} z & \left(z - \frac{1}{z}\right) q_n \\ z\, r_n & \frac{1}{z} + \left(z - \frac{1}{z}\right) q_n\, r_n \end{bmatrix},$$

$$\mathcal{T}_n = \begin{bmatrix} \frac{-i(z^2-1)[1+(z^2+1)\, q_{n-1}\, r_n]}{z^2(1+q_{n-1}\, r_n)} & \frac{i(z^2-1)q_{n-1}}{1+q_{n-1}\, r_n} - \frac{i(z^2-1)q_n}{z^2(1-q_n\, r_n)} \\ \frac{-ir_{n-1}}{1-q_{n-1}\, r_{n-1}} + \frac{i\, z^2\, r_n}{1+q_{n-1}\, r_n} & \frac{i(z^2-1)}{1+q_{n-1}\, r_n} \end{bmatrix}.$$

- DDNLS derived by imposing the compatibility condition

$$\dot{\mathcal{X}}_n + \mathcal{X}_n\, \mathcal{T}_{n+1} - \mathcal{T}_n\, \mathcal{X}_n = 0.$$

The linear system associated with DDNLS system

$$\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix} = \begin{bmatrix} z & \left(z-\dfrac{1}{z}\right) q_n \\ z\, r_n & \dfrac{1}{z} + \left(z - \dfrac{1}{z}\right) q_n\, r_n \end{bmatrix} \begin{bmatrix}\alpha_{n+1}\\ \beta_{n+1}\end{bmatrix}, \qquad n\in\mathbb{Z}, \quad t\in\mathbb{R}.$$

- n discrete independent variable
- z spectral parameter
- q_n and r_n complex-valued scalar quantities, potentials
- $\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}$ value of wavefunction at spacial location n

Tsuchida's formulation of the linear system for DDNLS

- Tsuchida (2002)

$$\begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} z - \left(z - \dfrac{1}{z}\right) q_n r_n & q_n \\ \left(-1 + \dfrac{1}{z^2}\right) r_n & \dfrac{1}{z} \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

- Tsuchida (2012)

$$\begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} = \begin{bmatrix} z & \left(z - \dfrac{1}{z}\right) q_n \\ z\, r_n & \dfrac{1}{z} + \left(z - \dfrac{1}{z}\right) q_n r_n \end{bmatrix} \begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix}, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

Contrast with DNLS system, continuous version

- DNLS system

$$\begin{cases} iq_t + q_{xx} - i(q^2 r)_x = 0, \\ ir_t - r_{xx} - i(q\, r^2)_x = 0, \end{cases} \qquad x, t \in \mathbb{R}.$$

  - $q = q(x,t)$ and $r = r(x,t)$

- linear system associated with the DNLS system

$$\frac{d}{dx}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix} -i\lambda & \sqrt{\lambda}\, q(x,t) \\ \sqrt{\lambda}\, r(x,t) & i\lambda \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}, \qquad x, t \in \mathbb{R}.$$

  - λ spectral parameter
  - x spacial coordinate, t time
  - $\sqrt{\lambda}\, q(x)$, $\sqrt{\lambda}\, r(x)$ energy-dependent potentials

The Marchenko method to solve integrable systems

- Inverse Scattering Transform

$$
\begin{array}{ccccc}
V(x,0) & \xrightarrow{\text{direct scattering at } t=0} & S(k,0) & \xrightarrow{\text{Fourier transform at } t=0} & F(y,0) \\
\text{solution} \big\downarrow & & \big\downarrow \text{time evolution} & & \big\downarrow \text{time evolution} \\
V(x,t) & \xleftarrow[\text{inverse scattering at time } t]{} & S(k,t) & \xleftarrow[\text{inverse Fourier transform at time } t]{} & F(y,t)
\end{array}
$$

- recover the potential from the solution to the Marchenko system

$$
\begin{array}{ccccc}
V(x,0) & \xrightarrow{\text{Marchenko kernel at } t=0} & F(y,0) & \xrightarrow{\text{time evolution}} & F(y,t) \\
\text{solution} \big\downarrow & & & & \big\downarrow \text{input to Marchenko system} \\
V(x,t) & \xleftarrow[\text{recover the potential}]{} & K(x,x^+,t) & \xleftarrow[\text{recover the potential}]{} & K(x,y,t)
\end{array}
$$

The Marchenko system for DDNLS

$$\begin{bmatrix}\bar{M}_{nm} & M_{nm}\end{bmatrix} + \begin{bmatrix}0 & \bar{\Omega}_{n+m} \\ \Omega_{n+m} & 0\end{bmatrix} + \sum_{l=n+1}^{\infty} \begin{bmatrix}\bar{M}_{nl} & M_{nl}\end{bmatrix} \begin{bmatrix}0 & \bar{\Omega}_{l+m} \\ \Omega_{l+m} & 0\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}, \qquad m > n, \quad t \in \mathbb{R}.$$

- reflection coefficients $R(z,t)$ and $\bar{R}(z,t)$
  - $R(z,t) = R(z,0)\, e^{-it(z-z^{-1})^2}, \qquad \bar{R}(z,t) = \bar{R}(z,0)\, e^{it(z-z^{-1})^2}.$
  - $\hat{R}_k(t) := \frac{1}{2\pi i} \oint dz\, R(z,t)\, z^{k-1}, \qquad \hat{\bar{R}}_k(t) := \frac{1}{2\pi i} \oint dz\, \bar{R}(z,t)\, z^{-k-1}.$
- bound-state data via the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$
  - $C(t) = C(0)\, e^{-it(A-A^{-1})^2}, \qquad \bar{C}(t) = \bar{C}(0)\, e^{it[\bar{A}-(\bar{A})^{-1}]^2}.$

$$\begin{cases} \Omega_k := \hat{R}_k + CA^{k-1}B, \qquad \bar{\Omega}_k := \hat{\bar{R}}_k + \bar{C}(\bar{A})^{-k-1}\bar{B}, \qquad k \text{ even}, \\ \Omega_k := 0, \quad \bar{\Omega}_k := 0, \qquad k \text{ odd}. \end{cases}$$

The matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$

$$A := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{bmatrix}, \quad B := \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & C_2 & \cdots & C_N \end{bmatrix},$$

$$A_j := \begin{bmatrix} z_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & z_j & 1 & \cdots & 0 & 0 \\ 0 & 0 & z_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_j & 1 \\ 0 & 0 & 0 & \dots & 0 & z_j \end{bmatrix}, \quad B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C_j := \begin{bmatrix} c_{j(m_j-1)} & c_{j(m_j-2)} & \cdots & c_{j1} & c_{j0} \end{bmatrix}.$$

- all z_j for $1 \leq j \leq N$ are inside the unit circle $|z| = 1$.
- each z_j has multiplicity m_j.

The matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$,

$$\bar{A} := \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A}_{\bar{N}} \end{bmatrix}, \quad \bar{B} := \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \vdots \\ \bar{B}_{\bar{N}} \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \cdots & \bar{C}_{\bar{N}} \end{bmatrix},$$

$$\bar{A}_j := \begin{bmatrix} \bar{z}_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{z}_j & 1 & \cdots & 0 & 0 \\ 0 & 0 & \bar{z}_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{z}_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \bar{z}_j \end{bmatrix}, \quad \bar{B}_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{C}_j := \begin{bmatrix} \bar{c}_{j(\bar{m}_j-1)} & \bar{c}_{j(\bar{m}_j-2)} & \cdots & \bar{c}_{j1} & \bar{c}_{j0} \end{bmatrix},$$

- all $\bar{z}_j$ for $1 \leq j \leq \bar{N}$ are outside the unit circle $|z| = 1$,
- each $\bar{z}_j$ has multiplicity $\bar{m}_j$,

The Marchenko system for DDNLS, the uncoupled version

$$\begin{cases} \left[M_{nm}\right]_1 + \bar{\Omega}_{n+m} - \sum_{l=n+1}^{\infty} \sum_{j=n+1}^{\infty} \left[M_{nj}\right]_1 \Omega_{j+l}\, \bar{\Omega}_{l+m} = 0, & m > n, \quad t \in \mathbb{R}, \\ \left[\bar{M}_{nm}\right]_2 + \Omega_{n+m} - \sum_{l=n+1}^{\infty} \sum_{j=n+1}^{\infty} \left[\bar{M}_{nj}\right]_2 \bar{\Omega}_{j+l}\, \Omega_{l+m} = 0, & m > n, \quad t \in \mathbb{R}, \end{cases}$$

$$\begin{cases} \left[\bar{M}_{nm}\right]_1 = -\sum_{l=n+1}^{\infty} \left[M_{nl}\right]_1 \Omega_{l+m}, & m > n, \quad t \in \mathbb{R}, \\ \left[M_{nm}\right]_2 = -\sum_{l=n+1}^{\infty} \left[\bar{M}_{nl}\right]_2 \bar{\Omega}_{l+m}, & m > n, \quad t \in \mathbb{R}. \end{cases}$$

- $[\cdot]_1$ and $[\cdot]_2$ denote the first and second components of the relevant column vectors

Recovery of potentials q_n and r_n

$$q_n = \frac{\displaystyle\sum_{l=n}^{\infty} [M_{nl}]_1 \sum_{k=n}^{\infty} [M_{nk}]_2}{\displaystyle\sum_{l=n}^{\infty} [\bar{M}_{nl}]_1 \sum_{k=n}^{\infty} [M_{nk}]_2 - \sum_{l=n}^{\infty} [M_{nl}]_1 \sum_{k=n}^{\infty} [\bar{M}_{nk}]_2}, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R},$$

$$r_n = \frac{\displaystyle\sum_{l=n-1}^{\infty} [\bar{M}_{(n-1)l}]_2}{\displaystyle\sum_{l=n-1}^{\infty} [M_{(n-1)l}]_2} - \frac{\displaystyle\sum_{l=n}^{\infty} [\bar{M}_{nl}]_2}{\displaystyle\sum_{l=n}^{\infty} [M_{nl}]_2}, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

Contrast with the Marchenko method for DNLS system

- linear system of Marchenko integral equations

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y,t) & K_1(x,y,t) \\ \bar{K}_2(x,y,t) & K_2(x,y,t) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y,t) \\ \Omega(x+y,t) & 0 \end{bmatrix}$$

$$+ \int_x^\infty dz \begin{bmatrix} -iK_1(x,z,t)\,\Omega'(z+y,t) & \bar{K}_1(x,z,t)\,\bar{\Omega}(z+y,t) \\ K_2(x,z,t)\,\Omega(z+y,t) & i\bar{K}_2(x,z,t)\,\bar{\Omega}'(z+y,t) \end{bmatrix}, \qquad x < y.$$

- reflection coefficients $R(\sqrt{\lambda},0)$ and $\bar{R}(\sqrt{\lambda},0)$
- bound-state data via the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$

$$\begin{cases} \Omega(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \frac{R(\sqrt{\lambda},0)}{\sqrt{\lambda}}\, e^{4i\lambda^2 t} e^{i\lambda y} + C\, e^{4iA^2 t} e^{iAy}\, B, \\ \bar{\Omega}(y,t) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\lambda\, \frac{\bar{R}(\sqrt{\lambda},0)}{\sqrt{\lambda}}\, e^{-4i\lambda^2 t} e^{-i\lambda y} + \bar{C}\, e^{-4i\bar{A}^2 t} e^{-i\bar{A}y}\, \bar{B}. \end{cases}$$

Contrast with recovery of potentials in the DNLS system

$$\begin{cases} q(x,t) = -2K_1(x,x,t)\exp\left(-4\int_x^\infty dz\left[\bar{K}_1(z,z,t) - K_2(z,z,t)\right]\right), \\ r(x,t) = -2\bar{K}_2(x,x,t)\exp\left(4\int_x^\infty dz\left[\bar{K}_1(z,z,t) - K_2(z,z,t)\right]\right). \end{cases}$$

Reflectionless case: explicit solutions for DDNLS system

- reflectionless case: separable-kernel Marchenko system and hence explicit solutions
- closed-form, compact formulas for explicit solutions involving matrix exponentials
- "unpacking" matrix exponentials yields explicit solutions in terms of elementary functions
- use (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as input to the Marchenko system
- obtain the Marchenko solution $[M_{nm}]_1$, $[M_{nm}]_2$, $[\bar{M}_{nm}]_1$, and $[\bar{M}_{nm}]_2$
- obtain q_n and r_n

Explicit solutions using (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$

- construct $\mathcal{E}$, $\bar{\mathcal{E}}$, Υ, and $\bar{\Upsilon}$ via

$$\mathcal{E} := e^{-it(A-A^{-1})^2}, \quad \bar{\mathcal{E}} := e^{it[\bar{A}-(\bar{A})^{-1}]^2}, \quad \Upsilon := \sum_{k=0}^{\infty} A^k\, B\, \bar{C}\, (\bar{A})^{-k}, \quad \bar{\Upsilon} := \sum_{k=0}^{\infty} (\bar{A})^{-k}\, \bar{B}\, C\, A^k.$$

- construct U_n and $\bar{U}_n$ via

$$U_n := I - \bar{\mathcal{E}}\, (\bar{A})^{-n-2}\, \bar{\Upsilon}\, \mathcal{E}\, A^{2n+1}\, \Upsilon\, (\bar{A})^{-n-1}, \quad \bar{U}_n := I - \mathcal{E} A^n \Upsilon \bar{\mathcal{E}} (\bar{A})^{-2n-3}\, \bar{\Upsilon}\, A^{n+1}.$$

- construct $\left[M_{nm}\right]_1$, $\left[M_{nm}\right]_2$, $\left[\bar{M}_{nm}\right]_1$, and $\left[\bar{M}_{nm}\right]_2$ via

$$\begin{cases} \left[M_{nm}\right]_1 = -\bar{C}\, (\bar{A})^{-n}\, (U_n)^{-1}\, \bar{\mathcal{E}}\, (\bar{A})^{-m-1}\, \bar{B}, \\ \left[M_{nm}\right]_2 = C\, A^n\, (\bar{U}_n)^{-1}\, \mathcal{E}\, A^n\, \Upsilon\, \bar{\mathcal{E}}\, (\bar{A})^{-n-m-2}\, \bar{B}, \\ \left[\bar{M}_{nm}\right]_1 = \bar{C}\, (\bar{A})^{-n}\, (U_n)^{-1}\, \bar{\mathcal{E}}\, (\bar{A})^{-n-2}\, \bar{\Upsilon}\, \mathcal{E}\, (A)^{n+m}\, B, \\ \left[\bar{M}_{nm}\right]_2 = -C\, A^n\, (\bar{U}_n)^{-1}\, \mathcal{E}\, A^{m-1}\, B. \end{cases}$$

- obtain q_n and r_n explicitly in terms of (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$.

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