Scattering structure of linearized Vlasov-Poisson equations (and more)

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thanks to F. Charles, R. Weder, A. Rege, C. Buet and V. Fournet

Analogies between linear Vlasov-Poisson and Scattering theory

P. Degond : Spectral theory of the linearized Vlasov-Poisson equation. Trans. Amer. Math. Soc. 294 (1986)

Expansion (41) was the aim of this paper. However, expansion (40) presents interesting analogies with Lax and Phillips' scattering theory, which we detail in the conclusion.

3.4. *Comparison between Vlasov's equation and scattering theory*. One of the aims of the scattering theory [**2**] is to obtain an asymptotic behaviour for the wave equation outside a bounded obstacle. Although the total energy of the solution remains constant, the dispersion towards infinity leads to a local decay of the L^∞ norm of the solution. This decay is well expressed by expansion (6).

Vlasov-Poisson equations

• 1D-1V non linear Vlasov-Poisson equations for electrons, with given ion density

$$\begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & t > 0,\ x \in \mathbb{T},\ v \in \mathbb{R}, \\ \partial_x E = \rho_{\mathrm{ions}}(x) - \int_{\mathbb{R}} f dv, & t > 0,\ x \in \mathbb{T}. \end{cases}$$

with unknowns $f(t,x,v) \geq 0$ and $E(t,x) = -\partial_x \varphi(t,x)$.

Note that f is constant along the characteristics

$$\dot{x} = v \text{ and } \dot{v} = -E \Longrightarrow \dot{f} = 0.$$

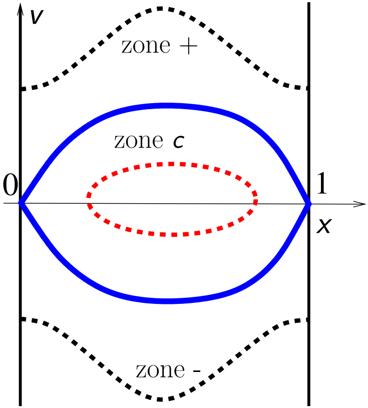
• Non homogeneous stationary states ($\not\subset$ Degond 86' or Mouhot-Villani 11') + linearization

$$\begin{cases} f_0(x,v) = n_0(x) e^{-\frac{v^2}{2}}, \quad n_0(x) = e^{\varphi_0(x)}, & \text{(BGK/Sagdeev/...)}, \\ E_0(x) = -\varphi_0'(x), \quad -\varphi_0''(x) + 2\pi e^{\varphi_0(x)} = \rho_{\mathrm{ions}}(x), & \end{cases}$$

Characteristics 1D-1V

- Free transport : $v\partial_x + \varphi_0'(x)\partial_v$
- Characteristic lines :
$x' = v$ and $v' = \varphi_0'(x)$
- Invariants : $\frac{v^2}{2} - \varphi_0(x) = H$.

Here φ_0 is **one bump**,
increases from 0 to x^*, decreases from x^* to 1,
and by convention
$\varphi_0 \geq 0 = \varphi_0(0) = \varphi_0(1)$.



The central zone of trapped particles : electron hole.

- Linearize around this family of stationary solutions

$$\begin{cases} f(t,x,v) = f_0(x,v) + \varepsilon\sqrt{f_0(x,v)}u(t,x,v) + O(\varepsilon^2), \\ E(t,x) = E_0(x) + \varepsilon F(t,x) + O(\varepsilon^2). \end{cases}$$

- One gets the 1D-1V linearized Vlasov-Poisson equation

$$\begin{cases} \partial_t u + v\partial_x u - E_0(x)\partial_v u + F(u)v\sqrt{f_0(x,v)} = 0, \\ F(u) = -(\partial_x)^{-1}\int_{\mathbb{R}} u\sqrt{f_0(x,v)}dv. \end{cases}$$

Field-particles coupling is provided by the operator

$$u \longmapsto -F(u)v\sqrt{f_0}$$

which has **no specific structure** (neither symmetric nor anti-symmetric).

Quadratic energy estimates

For convenience set $M(x,v) = \sqrt{f_0(x,v)} = e^{-\frac{v^2}{4}+\frac{\varphi_0(x)}{2}}$ and reformulate as the **Vlasov-Ampère equations**

$$\begin{cases} \partial_t u + v\partial_x u - E_0 \partial_v u & = -vMF, \qquad t>0, \\ \partial_t F & = \int_{\mathbb{R}} uvMdv, \quad t>0. \end{cases}$$

Now the right-hand side coupling terms is **anti-symmetric**.
One has the energy identity which already shows linear stability

$$\frac{d}{dt}\left(\int \int u^2 dvdx + \int F^2 dx \right) = 0$$

To be rigorous, define $U=(u,F) \in X := L^2(\mathbb{T}\times\mathbb{R}) \times L^2_0(\mathbb{T})$. One has

$$U'(t) = \mathbf{i}HU(t), \qquad H^* = H,$$

where $H = H_0 + K$ (perturbation K **compact in v, but not compact in x**)

$$\mathbf{i}H_0 = \left(\begin{array}{c|c} -v\partial_x + E_0\partial_v & 0 \\ \hline 0 & 0 \end{array} \right) \text{ and } \mathbf{i}K = \left(\begin{array}{c|c} 0 & -vM \\ \hline 1^* \int_v vM & 0 \end{array} \right)$$

The Gauss law is propagated by the dynamics

$$U(0) \in \left\{ (u,F) \in X \mid \int_{\mathbb{R}} uMdv + \partial_x F = 0 \right\} \subset \ker H.$$

Classical scattering theory

- Reed-Simon Methods of modern mathematical physics : III Scattering theory, 79'.
- Kato Perturbation Theory for Linear Operators, 80'.
- Lax, Functional analysis, 2002.

Take X Hilbert space, and two closed self-adjoint operators $H_0^* = H_0$ and $K^* = K$

$$U'(t) = iHU(t) \text{ with } H = H_0 + K.$$

Typically H_0 is unbounded but "simple" and K is "small".

- Goal : for $t \to \pm\infty$, compare $U'(t) = e^{iHt}U_0$ with $\widetilde{U}'(t) = e^{iH_0 t}\widetilde{U}_0$.

Some possible tools

a) explicit calculation of the spectrum.

b) Møller operators and trace class perturbation.

c) the Lipmann-Schwinger equation.

Vlasov-Poisson is "almost" trace-class

Take $E_0 := 0$ (the homogeneous case).

$$\begin{cases} \partial_t u + v\partial_x u & = -ve^{-v^2/4}F, \qquad t>0, \\ \partial_t F & = \int_{\mathbb{R}} uve^{-v^2/4}dv, \quad t>0. \end{cases}$$

Lemma

Take $n \in \mathbb{N}^*$ and $z = i\beta$ with $\beta \in \mathbb{R}^*$:
the operator $T = (H-z)^{-n} - (H_0 - z)^{-n}$ is not trace class.

The Fourier decomposition is $T = \bigoplus_{k\in\mathbb{Z}} e^{ikx} T_k$ with T_k defined by

$$H_0^k = \left(\begin{array}{c|c} vk & 0 \\ \hline 0 & 0 \end{array}\right) \text{ and } H^k = \left(\begin{array}{c|c} vk & -ive^{-\frac{v^2}{4}} \\ \hline i\int ve^{-\frac{v^2}{4}}\cdot dv & 0 \end{array}\right)$$

The difference is finite rank, so T_k **is trace-class individually**.

Let $u_k = \left(ie^{-\frac{v^2}{4}}, k\right)$ be the zero eigenvector $H_k u_k = 0$

$$\lim_{|k|\to\infty}\left(k^2|z|^{2n}\frac{\|T^k u_k\|^2}{\|u_k\|^2}\right) = \sqrt{2\pi} \Longrightarrow \|T\|_1 \gtrsim \frac{(2\pi)^{\frac14}}{|z|^n}\sum_{k\neq 0}\frac{1}{|k|} = +\infty.$$

Then the full operator T is "almost" trace-class.

Morrison integral operator= generalized Moller wave operator

Here $E_0 = \varphi_0 = 0$ and u satisfies $\partial_t u + v\partial_x u + vME = 0$ with the Gauss law.

Consider the integro-differential operator L (rewriting of Morrison $\mathcal{G}$ transform)

$$Lu = (-\partial_{xx} + q(v))\, u(x,v) - vM(v)P.V. \int_{\mathbb{R}} \frac{1}{w-v} u(x,w)M(w)dw$$

where the function q is defined by

$$q(v) = P.V. \int_{\mathbb{R}} \frac{w}{w-v} M(w)^2 dw.$$

Then $h = Lu$ is a solution of free transport $\partial_t h + v\partial_x h = 0$.

- Morrison, Hamiltonian Description of Vlasov Dynamics : Action-Angle Variables for the Continuous Spectrum, 2000
- Després, Symmetrization of Vlasov-Poisson equations, 2014.

In other words $u(t) = L^{-1} e^{-v\partial_x t} L u_0$, that is

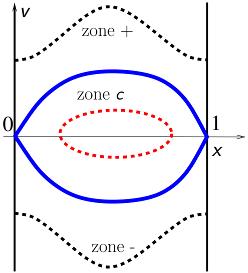
$$e^{iHt} = L^{-1} e^{iH_0 t} L, \quad iH_0 = -v\partial_x.$$

Intertwinning property $LH = H_0 L$: **generalized wave operators, Yafaev 01**

Model

$$\begin{cases} \partial_t u + v\partial_x u - E_0\partial_v u & = -vMF, \qquad t>0, \quad (x,v)\in I\times\mathbb{R}, \\ \partial_t F & = \int_{\mathbb{R}} uvMdv, \quad t>0, \quad x\in I, \end{cases}$$

with $E_0(x) = -\varphi_0'(x)$ and φ_0 regular with any number of bumps.



Characteristic lines $\frac{1}{2}v^2 + \varphi_0(x) = \text{cst}$ for a one bump electric field.

Technical tool : hiding the electric field

- Remind $M(x,v)=\sqrt{f_0(x,v)}=e^{-\frac{v^2}{4}+\frac{\varphi_0(x)}{2}}$. Make the change of unknown

$$w(x,v,t)=u(x,v,t)+\gamma(x)M(x,v)F(x,t),$$

where the weight $\gamma(x)$ is such that the total energy is preserved
$\int_{\mathbb{T}\times\mathbb{R}} w^2 dvdx=\int_{\mathbb{T}\times\mathbb{R}} u^2 dvdx+\int_{\mathbb{T}} F^2$.

- It holds for γ solution of the Ricatti equation

$$\gamma'(x)+\alpha^2\gamma(x)^2 e^{\varphi_0(x)}=1, \qquad \alpha=(2\pi)^{\frac{1}{4}}, \qquad x\in\mathbb{T}=[0,1]_{\mathrm{per}}.$$

Indeed

$$\begin{aligned}\|w\|_{xv}^2 &= \|u\|_{xv}^2+2\int_{\mathbb{T}\times\mathbb{R}} u\gamma MF+\int_{\mathbb{T}\times\mathbb{R}}\gamma^2M^2F^2\\ &=\|u\|_{xv}^2-2\int_{\mathbb{T}}\gamma F\partial_x F+\int_{\mathbb{T}}\alpha^2\gamma^2 e^{\varphi_0(x)}F^2\\ &=\|u\|_{xv}^2+\int_{\mathbb{T}}\left(\gamma'+\alpha^2\gamma^2e^{\varphi_0(x)}\right)F^2\end{aligned}$$

- Then w satisfies an autonomous equation **without Gauss, Poisson or Ampère**

$$\partial_t w = i\mathcal{H}w, \quad \mathcal{H} = i\mathcal{H}_0 + i\mathcal{K}, \quad i\mathcal{K} = i\mathcal{K}_1 + i\mathcal{K}_2,$$

where $i\mathcal{H}_0 = -(v\partial_x w - E_0(x)\partial_v)$ is the free transport operator,

$$i\mathcal{K}_1 w = \gamma \left(vM \int_{\mathbb{R}} wMdv - M \int_{\mathbb{R}} wvMdv \right)$$

and

$$i\mathcal{K}_2 w = \gamma M \int_{\mathbb{T}\times\mathbb{R}} wvMdv - \left(\int_{\mathbb{T}\times\mathbb{R}} w\gamma M \right) Mv.$$

- By construction $\mathcal{H}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$ are self-adjoint in $L^2(\mathbb{T}\times\mathbb{R})$.
- **$\mathcal{K}_1$ regularizes in v, not in x.**
- $\mathcal{K}_2$ is finite rank.
- One has $\mathrm{Ker}(\mathcal{K}) = \underset{n\geq 2}{\mathrm{Span}} \{a_n(x)\psi_n(v)\}$, $\quad a_n \in L^2(\mathbb{T})$ where $(\psi_n)_{n\in\mathbb{N}}$ be the orthonormal $\int_{\mathbb{R}} \psi_n(v)\psi_m(v)dv = \delta_{nm}$ complete family of Hermite functions

$$\psi_0(v) = \exp(-v^2/4)/\alpha, \quad \psi_1(v) = v\exp(-v^2/4)/\alpha, \quad \psi_2(v) = \dots$$

- It is sufficient to show that $\mathcal{T}$ is trace class

$$\mathcal{T} = (\mathcal{H}_0 - z)^{-1}\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}$$

Integral operators

One has $\mathcal{T}^*\mathcal{T} = (\mathcal{H}_0 - \bar{z})^{-1}\mathcal{K}_1(\mathcal{H}_0^2 + |z|^2)^{-1}\mathcal{K}_1(\mathcal{H}_0 - z)^{-1}$, where $z = i\beta \in i\mathbb{R}^*$.

Lemma

One has $\mathrm{Ker}(\mathcal{T}^*\mathcal{T}) = \mathrm{Span}_{n\geq 2}\left\{(\mathcal{H}_0 - z)a_n\psi_n, a_n \in H^1(\mathbb{T})\right\}$.

Lemma

Let $\lambda \in \mathbb{R}^*$. The equation $(\mathcal{T}^*\mathcal{T})w = \lambda w$ for $w \neq 0$ is equivalent to two decoupled integral equations

$$\begin{pmatrix} \gamma e^{\varphi_0}\mathcal{T}_2\gamma e^{\varphi_0}\mathcal{T}_1 a \\ \gamma e^{\varphi_0}\mathcal{T}_1\gamma e^{\varphi_0}\mathcal{T}_2 b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \quad (a,b) \neq (0,0), \tag{1}$$

where $\mathcal{T}_1, \mathcal{T}_2 : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ are integral operators

$$\begin{cases} \mathcal{T}_1 a(x) = \alpha^2 \int_{\mathbb{R}} \psi_0(v)\left[(\mathcal{H}_0^2 + |z|^2)^{-1}(a\psi_0)\right](x,v)dv, \\ \mathcal{T}_2 b(x) = \alpha^2 \int_{\mathbb{R}} \psi_1(v)\left[(\mathcal{H}_0^2 + |z|^2)^{-1}(b\psi_1)\right](x,v)dv. \end{cases} \tag{2}$$

Moreover $\mathcal{T}_1$ and $\mathcal{T}_2$ are self adjoint, bounded, positive and injective.

Hint : take $w = (\mathcal{H}_0 - \bar{z})^{-1}\left(a(x)\psi_0(v) + b(x)\psi_1(v)\right) \in \mathrm{Ker}(\mathcal{T}^*\mathcal{T})^{\perp}$.

$\mathcal{T}_2$

Lemma

Assume $E_0 \in W^{s+1,\infty}(\mathbb{T})$ for $s \geq 0$. Then there exists $C_s > 0$ such that $\|\mathcal{T}_2 b\|_{H^{s+2}(\mathbb{T})} \leq C_s \|b\|_{H^s(\mathbb{T})}$.

Hint of the proof : with over-simplification, take $E_0 = 0$ (and $z = i$). Then

$$\mathcal{T}_2 b(x) = C \int_{\mathbb{R}} v e^{-\frac{v^2}{4}} \left[\left(-v^2 \partial_{xx} + 1\right)^{-1} v e^{-\frac{v^2}{4}} b \right] (x, v) dv$$

$$\approx \widetilde{\mathcal{T}}_2 b(x) = C \int_{\mathbb{R}} v^2 \left[\left(-v^2 \partial_{xx} + 1\right)^{-1} e^{-\frac{v^2}{2}} b \right] (x, v) dv$$

Clearly

$$\partial_{xx} \widetilde{\mathcal{T}}_2 b(x) = C \int_{\mathbb{R}} \left[\left(v^2 \partial_{xx}\right) \left(-v^2 \partial_{xx} + 1\right)^{-1} \right] e^{-\frac{v^2}{2}} b(x, v) dv$$

so $\|\widetilde{\mathcal{T}}_2 b\|_{H^{s+2}(\mathbb{T})} \leq C_s \|b\|_{H^s(\mathbb{T})}$.

One checks it is the same for operator $\mathcal{T}_2$.

$\mathcal{T}_1$

Lemma

Under the same assumptions, there exists a constant $C_s > 0$ such that $\|\mathcal{T}_1 a\|_{H^{s+\frac{1}{4}}(\mathbb{T})} \leq C_s \|a\|_{H^s(\mathbb{T})}$.

Hint of the proof : same over-simplification ($E_0 = 0$) and $z = i$. Then

$$\mathcal{T}_1 a(x) = C \int_{\mathbb{R}} e^{-\frac{v^2}{4}} \left[\left(-v^2 \partial_{xx} + 1\right)^{-1} e^{-\frac{v^2}{4}} b \right] (x, v) dv$$

$$= C \int_{\mathbb{R}} e^{-\frac{v^2}{4}} \left(v\partial_x + 1\right)^{-1} \underbrace{\left[\left(-v\partial_x + 1\right)^{-1} e^{-\frac{v^2}{4}} b \right]}_{=g} (x, v) dv$$

Extra-regularity comes from **compactness by integration.** (Golse-Sentis-Perthame-Lions).

Take $g = \left(-v\partial_x + 1\right)^{-1} e^{-\frac{v^2}{4}} b \in L^2(\mathbb{T} \times \mathbb{R})$,
and $f = \left(v\partial_x + 1\right)^{-1} g \in L^2(\mathbb{T} \times \mathbb{R})$. So

$$v\partial_x f = g - f \in L^2(\mathbb{T} \times \mathbb{R}).$$

One gets

$$\int_{\mathbb{R}} e^{-\frac{v^2}{4}} f(x, v) dx dv \in H^{\frac{1}{2}}(\mathbb{T}).$$

General case $E_0 \neq 0$

The general case is $g = (-v\partial_x + E_0(x)\partial_v + 1)^{-1}\, b \in L^2(\mathbb{T}\times\mathbb{R})$,
and $f = (v\partial_x - E_0(x)\partial_v + 1)^{-1}\, g \in L^2(\mathbb{T}\times\mathbb{R})$.

-Thanks to F. Golse.

So

$$v\partial_x f = g - f + \partial_v(E_0(x)f) \in L^2(\mathbb{T}; H^{-1}(\mathbb{R})).$$

With $m = 1$, one gets

$$\int_{\mathbb{R}} e^{-\frac{v^2}{4}} f(x,v)\,dxdv \in H^{\frac{1}{4}}(\mathbb{T}).$$

- Diperna and Lions, Global weak solutions of Vlasov-Maxwell systems, 1989.

Then the proof of the trace class property goes on.

Final statements 1/2

Theorem (Jour. Scat. Theory 2020, D.)

Assume the electric potential is smooth $\varphi_0 \in W^{4+\varepsilon,\infty}(\mathbb{T})$ with any number of bumps. Then the wave operators $\mathcal{W}_\pm(\mathcal{H},\mathcal{H}_0)$ exist and are complete. In particular one has the orthogonal decompositions between spaces associated to absolute continuous, singular continuous and discrete parts of the spectrum

$$L^2(\mathbb{T}\times\mathbb{R}) = \mathcal{X}_0^{\mathrm{ac}} \oplus \mathcal{X}_0^{\mathrm{sc}} \oplus \mathcal{X}_0^{\mathrm{pp}} = \mathcal{X}^{\mathrm{ac}} \oplus \mathcal{X}^{\mathrm{sc}} \oplus \mathcal{X}^{\mathrm{pp}} \tag{3}$$

and X^{ac} is in isometric bijection with $\mathcal{X}_0^{\mathrm{ac}}$.

Final statements 2/2

Hyp. 1 : The time needed for particles to travel along characteristics is a monotone function of the characteristic label : $\frac{d^2}{dx^2}\sqrt{\varphi_0^+ - \varphi_0(x)} < 0$
Hyp 2 : The initial data has zero mean value along the characteristics curves of the transport operator $v\partial_x - \varepsilon^2 E_0 \partial_v$ (same as Faou-et-al 2021).

Theorem (Ann. IHP 2019, D.)

Rescale $\varphi_0^\varepsilon(x) = \varepsilon\varphi_0(x)$ which has just one bump. For $0 < \varepsilon < \varepsilon_*$, then $\lim_{t\to\infty} \|F(t)\|_{L^\infty(I)} = 0$.

Vlasov-Poisson-Ampère nD

Now $\mathbf{x} \in \mathbb{R}^d$ with $d > 1$.

Start from

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{E}(t, \mathbf{x}) \cdot \nabla_{\mathbf{v}} f = 0, & t > 0, \\ \partial_t \mathbf{E} = \nabla \Delta^{-1} \nabla \cdot \int f \mathbf{v} dv, & t > 0, \end{cases}$$

with Gauss law/Poisson at initial time

$$\begin{cases} -\Delta \varphi = \rho_{\mathrm{ions}}(\mathbf{x}) - \int f dv, & t = 0 \\ \mathbf{E}(t, \mathbf{x}) = -\nabla_{\mathbf{x}} \varphi(t, \mathbf{x}), & t = 0. \end{cases}$$

Linearize

$$\begin{cases} f_0(\mathbf{x}, \mathbf{v}) = e^{-\frac{|\mathbf{v}|^2}{2} + \varphi_0(\mathbf{x})}, \\ f(t, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}) + \varepsilon \sqrt{f_0(\mathbf{x}, \mathbf{v})} u(t, \mathbf{x}, \mathbf{v}) + O(\varepsilon^2), \\ E(t, \mathbf{x}) = E_0(\mathbf{x}) + \varepsilon F(t, \mathbf{x}) + O(\varepsilon^2). \end{cases}$$

It yields

$$\begin{cases} \partial_t u + \mathbf{v} \cdot \nabla_{\mathbf{x}} u - \mathbf{E}_0 \cdot \nabla_{\mathbf{v}} u + \mathbf{F} \cdot \mathbf{v} \sqrt{f_0} = 0, & t > 0, \\ \partial_t \mathbf{F} = \nabla \Delta^{-1} \nabla \cdot \int u \mathbf{v} \sqrt{f_0} dv, & t > 0. \end{cases}$$

Set $w = u + i\sqrt{f_0}L\varphi$ where

$$L = (-\Delta)^{-\frac{1}{2}} \left[(-\Delta)^{-\frac{1}{2}} \alpha_0 (-\Delta)^{-\frac{1}{2}} \right]^{-\frac{1}{2}} (-\Delta)^{\frac{1}{2}}.$$

where the weight is $\alpha_0(\mathbf{x}) = \int_{\mathbb{R}^d} f_0(\mathbf{x}, \mathbf{v}) dv = (2\pi)^{\frac{d}{2}} n_0(\mathbf{x}) > 0$.

Lemma

One has

$$\|w\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} + \|F\|^2_{L^2(\mathbb{R}^d)}$$

with the autonomous equation

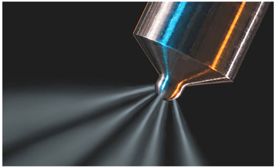
$$\partial_t w + \mathbf{v} \cdot \nabla_{\mathbf{x}} w - \mathbf{E}_0 \cdot \nabla_{\mathbf{v}} w - iAw = 0$$

where the symmetric operator A is defined by

$$Aw = \sqrt{f_0} L (-\Delta)^{-1} \nabla \cdot \int \sqrt{f_0} \mathbf{v} w dv - \sqrt{f_0} \mathbf{v} \cdot \nabla L (-\Delta)^{-1} \int \sqrt{f_0} w dv.$$

Sprays

Thin sprays :



(a) Diesel engine fuel injector

(b) Medical spray

Thick sprays :

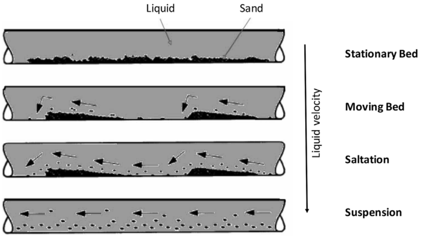
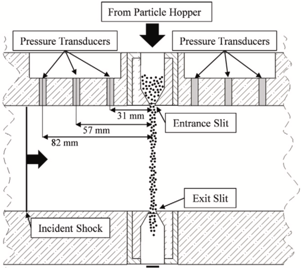


Fig. 1. Sand flow regime in horizontal pipelines.

(c) Leporini at al (2019)



(d) Daniel-Wagner (2022)

Barotropic fluid+particules model à la Desvillettes

With C. Buet (CEA) and V. Fournet (PHD CEA)

$$
\begin{cases}
\partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0,\\
\partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = -m_\star \int \mathbf{\Gamma} f dv,\\
\alpha = 1 - m_\star \int f dv,\\
m_\star\mathbf{\Gamma} = -m_\star\nabla p - D_\star(\mathbf{v}-\mathbf{u}),\\
\partial_t f + \mathbf{v}\cdot\nabla_x f + \nabla_v\cdot(\mathbf{\Gamma} f) = 0.
\end{cases}
$$

Fournet-Buet-D. : Analog of Linear Landau Damping in a coupled Vlasov-Euler system for thick sprays `https://cnrs.hal.science/LJLL/hal-04265990v1`.

Linearization

Linearize around $f_0(v) = e^{-v^2/2}$ (other profiles are possible) with $D_\star = 0$, $m_\star = 1$, $\mathbf{u}_0 = 0$, $\alpha_0 = cst$, ...,

$$\begin{cases} \partial_t \tau_1 = \nabla \cdot \mathbf{u}_1 + \nabla \cdot \int \sqrt{f_0}\, g_1\, \mathbf{v}\, dv, \\ \partial_t \mathbf{u}_1 = \nabla \tau_1, \\ \partial_t g_1 = -\mathbf{v} \cdot \nabla_x g_1 + \sqrt{f_0}\, \mathbf{v} \cdot \nabla \tau_1. \end{cases}$$

In 1D for Fourier mode k, it is rewritten as

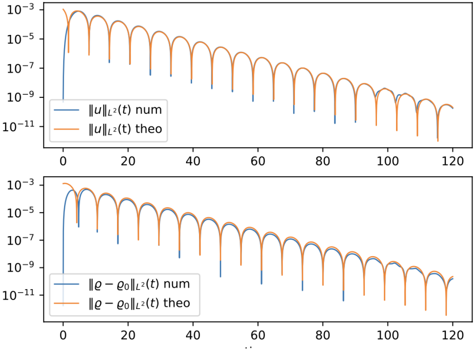
$$U'(t) = iHU(t) \text{ with } H = k \begin{pmatrix} 0 & 1 & \int e^{-\frac{v^2}{4}} \cdot v dv \\ 1 & 0 & 0 \\ e^{-\frac{v^2}{4}} v & 0 & -v \end{pmatrix}.$$

Prop : For $k \neq 0$, $X^{\mathrm{ac}} = X$.
So the acoustic energy tends to zero : $\tau_1(t)^2 + u_1(t)^2 \to 0$.

An interesting collective phenomenon

Linearize the equations around a solution at rest (nothing moves) with a Gaussian profile $f_0(v) = e^{-v^2/2}$ for the particles

Initial conditions:
$\varrho_0 = 1, \quad u_0 = 0, \quad f_0(x,v) = (1+\varepsilon\cos(kx))e^{-v^2/2}, \quad \varepsilon = 10^{-3}$
Orange curve $\propto e^{\Im m(\omega)t}\cos(\Re e(\omega)t)$, $\omega(k)$ solution of $\frac{k^2}{\omega^2} + \int \frac{f_0'(v)}{v-\omega/k} = 1$



From models to scattering

Focus on Vasov-Poisson 1D-1V non homogeneous.

Other problems

Specialists immediately notice the similarity with Linear Landau Damping in plasma physics

- First predicted by Landau[46'] for the linearized Vlasov-Poisson system

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_x f - \mathbf{E} \cdot \nabla_v f_0 = 0, \\ \nabla_x \cdot \mathbf{E} = -\int f \, \mathrm{d}v \end{cases}$$

around maxwellian equilibrium $f_0(v) = e^{-v^2/2}$

- Landau showed the damping of the electric field

$$\|\mathbf{E}(t)\| = \mathcal{O}\left(e^{\mathrm{Im}(\omega)t}\cos(\mathrm{Re}(\omega))\right),$$

with $\omega(k) \in \mathbb{C}$ verifies a dispersion relation

$$\int_{\mathbb{R}} \frac{\partial_v f_0(v)}{v - \omega/k} \, \mathrm{d}v = k^2.$$

- To show this, take the ansatz $f(t,x,v) = \alpha(v)e^{-i\omega t}e^{ikx}, \quad E(t,x) = \beta e^{-i\omega t}e^{ikx}.$

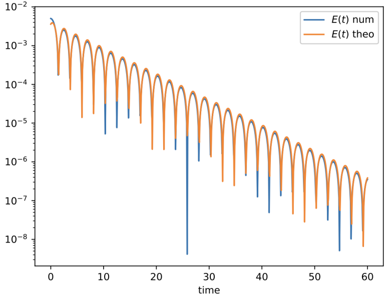


Figure 1: Landau damping for (nonlinear) Vlasov-Poisson

Rough explanation

Set all non important coefficients to 1 and annulate the friction $D_* = 0$.
The kinetic equation in Thick Sprays writes

$$\partial_t f + v\partial_x f + \Gamma \partial_v f = 0, \qquad \Gamma = -\partial_x p.$$

The Vlasov-Poisson equation in plasma physics writes

$$\partial_t f + v\partial_x f + E \partial_v f = 0, \qquad E = -\partial_x \phi.$$

This similarity is the key ingredient.
It explains why the mathematical developments for the derivation of the dispersion relation follow the same route.

Conclusion

• Kinetic equations coupled with macroscopic equations (Poisson-Ampère-Euler-...) in a non dissipative manner can be analyzed with classical Scattering Theory.

- It offers a simple explanation of Linear Landau Damping around homogeneous profiles which can now be seen as an exercice in Scattering Theory.
- It offers possibilities to explore Linear Landau Damping around non homogeneous profiles.
- There is an extension to magnetized Vlasov-Poisson-Ampère equations (Weder+Charles+Rege+D.).

• The non linear case (Mouhot-Villani-...) seems difficult to analyze within classical Scattering Theory.

• Same tools can be used for Thick Sprays (neutral particles + Euler equations).