THE FEYNMAN PROPAGATOR ON CURVED SPACETIMES

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On many curved spacetimes one can define four natural *Green functions* of the Klein-Gordon equation:

- the *retarded* or *forward propagator* $G^{\vee}$,
- the *advanced* or *backward propagator* $G^{\wedge}$,
- the (distinguished) *Feynman propagator* G^{F},
- the (distinguished) *antiFeynman propagator* $G^{\overline{\mathrm{F}}}$.

One also introduces various *bisolutions* of the Klein-Gordon equation: *positive/negative frequency 2-point functions* $G^{(\pm)}$ and the *Pauli-Jordan propagator* G^{PJ}.

They are key ingredients of perturbative *Quantum Field Theory*. I will discuss them from the point of view of operator theory.

I. FLAT SPACETIME.

Consider first the *Klein-Gordon equation* on the flat *Minkowski space* $\mathbb{R}^{1,d-1}$:

$$(-\Box + m^2)\psi = 0. \tag{1}$$

We will say that $G^\bullet(x,y)$ is a *Green function* of (1) if

$$(-\Box_x + m^2)G^\bullet(x,y) = (-\Box_y + m^2)G^\bullet(x,y) = \delta(x-y).$$

We will say that $G^\bullet(x,y)$ is a *bisolution* of (1) if

$$(-\Box_x + m^2)G^\bullet(x,y) = (-\Box_y + m^2)G^\bullet(x,y) = 0.$$

There are four Green functions invariant wrt the restricted Poincaré group:

- the *forward/backward propagator*

$$G^{\vee/\wedge}(x,y) \coloneqq \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2\pm \mathrm{i}0\,\mathrm{sgn}\,p_0}\,\mathrm{d}p,$$

- the *Feynman/anti-Feynman propagator*

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) \coloneqq \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2\mp \mathrm{i}0}\,\mathrm{d}p.$$

$G^{\vee}$ and $G^{\wedge}$ are related to the classical *Cauchy problem*, because their support is in the forward, resp. backward cone. G^{F} and $G^{\overline{\mathrm{F}}}$ are used in QFT to compute *Feynman diagrams*.

They satisfy the identity $G^{\mathrm{F}}+G^{\overline{\mathrm{F}}}=G^{\vee}+G^{\wedge}$.

Here are the most important bisolutions:

- the *Pauli–Jordan propagator* or *commutator function*
$$G^{\mathrm{PJ}}(x,y) := G^{\vee} - G^{\wedge},$$
- the *positive frequency* or *Wightman* 2-point function
$$G^{(+)}(x,y) := \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\wedge}) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\vee}),$$
- the *negative frequency* or *anti-Wightman* 2-point function
$$G^{(-)}(x,y) := \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\wedge}) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\vee}).$$

Jointly, these Green functions and bisolutions, well motivated by QFT, will be informally called *propagators*.

After *quantization*, we obtain an operator–valued distribution $\mathbb{R}^{1,d-1} \ni x \mapsto \hat{\psi}^*(x) = \hat{\psi}(x)^*$ satisfying the Klein-Gordon equation and commutation relations

$$(-\Box + m^2)\hat{\psi}^*(x) = 0,$$
$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x, y).$$

We also have a state $(\Omega| \cdot \Omega)$ such that

$$(\Omega\,|\,\hat{\psi}(x)\hat{\psi}^*(y)\Omega) = G^{(+)}(x, y),$$
$$(\Omega\,|\,\hat{\psi}^*(x)\hat{\psi}(y)\Omega) = G^{(-)}(x, y),$$

$$\big(\Omega\,\big|\,\mathrm{T}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = -\mathrm{i}G^{\mathrm{F}}(x, y),$$
$$\big(\Omega\,\big|\,\overline{\mathrm{T}}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = \mathrm{i}G^{\overline{\mathrm{F}}}(x, y).$$

There are two distinct operator-theoretic interpretations of propagators. The first is based on the Hilbert space $L^2(\mathbb{R}^{1,3})$:

(1) The Klein-Gordon operator $K = -\Box + m^2$ is *essentially self-adjoint* on $C_\mathrm{c}^\infty(\mathbb{R}^{1,3})$ in the sense of $L^2(\mathbb{R}^{1,3})$.

(2) For $s > \frac{1}{2}$, as an operator $\langle t\rangle^{-s}L^2(\mathbb{R}^{1,3}) \to \langle t\rangle^{s}L^2(\mathbb{R}^{1,3})$, the Feynman propagator is the *boundary value of the resolvent of the Klein-Gordon operator*:

$$\operatorname*{s-lim}_{\epsilon\searrow 0}(K \mp \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}/\overline{\mathrm{F}}}.$$

Here $\langle t\rangle$ denotes the so-called "Japanese bracket"

$$\langle t\rangle := \sqrt{1+t^2}.$$

The second operator-theoretic approach is based on the *Krein space* $\mathcal{W} := (m^2 - \Delta)^{-\frac{1}{4}} L^2(\mathbb{R}^3) \oplus (m^2 - \Delta)^{\frac{1}{4}} L^2(\mathbb{R}^3)$, describing *Cauchy data*. More important than its scalar product is the indefinite *Klein-Gordon charge form*

$$(w|v)_{\mathrm{KG}} := (w_1|v_2) + (w_2|v_1),$$

Krein space is a space with the topology of a Hilbert space and a distinguished Hermitian form $(\cdot|Q\cdot)$ such that there exist $S_\bullet$ where $S_\bullet^2 = \mathbb{1}$ and

$$(v|w)_\bullet = (v|QS_\bullet w) = (S_\bullet v|Qw)$$

is a scalar product compatible with the topology.

Set $\zeta\overleftrightarrow{\partial}\xi = \zeta\partial\xi - \partial\zeta\xi$. Note that for $\zeta, \xi \in \mathcal{W}$, we have $\partial_\mu(\zeta\overleftrightarrow{\partial}^\mu\xi) = 0$. The Klein-Gordon charge is the integral of the above current over any Cauchy surface, e.g.:

$$(\zeta|\xi)_{\mathrm{KG}} = \int \overline{\zeta(t,\vec{x})}\overleftrightarrow{\partial}^0\zeta(t,\vec{x})\,\mathrm{d}\vec{x}.$$

For a bisolution $G^\bullet(x,y)$, a linear operator $C^\bullet$ on $\mathcal{W}$ is defined by

$$\int G^\bullet(x,t,\vec{y})\overleftrightarrow{\partial}^0\zeta(t,\vec{y})\,\mathrm{d}\vec{y}$$

$G^\bullet$ is then called the *Klein-Gordon kernel* of $C^\bullet$.

Example: $G^{\mathrm{PJ}}(x,y)$ is the Klein-Gordon kernel of identity.

The Klein-Gordon equation can be rewritten as a 1st order evolution equation preserving the scalar product, and more importantly, the charge form:

$$\big(\partial_t + \mathrm{i}B\big)w = 0,$$

$$B := \begin{bmatrix} 0 & \mathbb{1} \\ m^2 - \Delta & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} u \\ \mathrm{i}\partial_t u \end{bmatrix}.$$

Introduce the projections onto *positive/negative frequency solutions* (or *particles/antiparticles*): $\Pi^{(\pm)} := \mathbb{1}_{\mathbb{R}_+}(\pm B)$. Then $G^{(\pm)}(x, y)$ are the Klein-Gordon kernels of $\pm\Pi^{(\pm)}$.

II. CURVED SPACETIMES.

Consider a curved spacetime M with the *metric tensor* $g_{\mu\nu}$. Define the *d'Alembertian* and the *Klein-Gordon operator*

$$-\Box := -|g|^{-\frac{1}{2}}\partial_\mu |g|^{\frac{1}{2}} g^{\mu\nu}\partial_\nu, \qquad K := -\Box + m^2.$$

(One could also replace the term m^2 with an x-dependent *scalar potential*). How to generalize the well-known propagators from $\mathbb{R}^{1,d-1}$ to generic spacetimes?

We will restrict ourselves to *globally hyperbolic* spacetimes, that is Lorentzian manifolds possessing Cauchy surfaces and with time flowing forever.

As is well-known, if M is globally hyperbolic, then the *forward/backward propagators* have natural generalizations. Namely, there exist unique distributions $G^\vee$ and $G^\wedge$ such that

$$\begin{aligned}(-\Box + m^2)\zeta^{\vee/\wedge} &= f,\\ \operatorname{supp}\zeta^{\vee/\wedge} &\subset \text{future/past shadow of} \operatorname{supp} f\end{aligned}$$

is uniquely solved by

$$\zeta^{\vee/\wedge}(x) := \int G^{\vee/\wedge}(x,y) f(y) |g|^{\frac{1}{2}}(y)\, \mathrm{d}y.$$

Note that $-\Box$ is obviously *Hermitian* (symmetric) on $C_{\rm c}^\infty(M)$ in the sense of the Hilbert space $L^2(M,|g|^{\frac12})$. Assume it is *essentially self-adjoint*. Then its resolvent $(-\Box+m^2)^{-1}$ is well defined for complex m^2. For real m^2, not eigenvalues of $\Box$, we define the *operator-theoretic Feynman/antiFeynman propagator* as the integral kernel of

$$G_{\rm op}^{\rm F} := \lim_{\epsilon\searrow0}\frac{1}{(-\Box+m^2-{\rm i}\epsilon)},\quad G_{\rm op}^{\overline{\rm F}} := \lim_{\epsilon\searrow0}\frac{1}{(-\Box+m^2+{\rm i}\epsilon)}.$$

I believe that the following argument justifies this definition. Here is an elementary fact about *Fresnel integrals* (with $x\in\mathbb{R}$):

$$\frac{\int {\rm e}^{\pm{\rm i}(\frac{c}{2}x^2+Jx)}\,{\rm d}x}{\int {\rm e}^{\pm{\rm i}\frac{c}{2}x^2}\,{\rm d}x}=\exp\Big(\mp\frac{{\rm i}J^2}{2(c\pm{\rm i}0)}\Big).$$

If we use *path integrals*, the generating function formally is

$$Z(J) := \frac{\int \mathrm{e}^{\mathrm{i}S(\psi,\psi^*)+\mathrm{i}\psi J^*+\mathrm{i}\psi^* J}\mathcal{D}\psi\mathcal{D}\psi^*}{\int \mathrm{e}^{\mathrm{i}S(\psi,\psi^*)}\mathcal{D}\psi\mathcal{D}\psi^*}.$$

If the action is *quadratic*

$$\begin{aligned} S(\psi,\psi^*) &= -\int \big(\partial_\mu\psi^*(x)\partial^\mu\psi(x) + m^2\psi^*(x)\psi(x)\big)\sqrt{|g|}(x)\,\mathrm{d}x \\ &= -\,\big(\psi|(-\Box + m^2)\psi\big), \end{aligned}$$

then the path integral can be *rigorously defined* as

$$\begin{aligned} Z(J) &= \exp\Big(\mathrm{i}\int\int \overline{J(x)}G^{\mathrm{F}}_{\mathrm{op}}(x,y)J(y)\sqrt{|g|}(x)\sqrt{|g|}(y)\,\mathrm{d}x\,\mathrm{d}y\Big) \\ &= \exp\mathrm{i}\big(J|(-\Box + m^2 - \mathrm{i}0)^{-1}J\big). \end{aligned}$$

Essential self-adjointness of the d'Alembertian is easy in some special cases:

- *stationary* spacetimes;
- *Friedmann-Lemaitre-Robertson-Walker* (FLRW) spacetimes;
- $1+0$-*dimensional* spacetimes;
- *deSitter* and (the universal covering of) *anti-deSitter spacetime*, (which follows from general properties of symmetric spaces).

On a class of *asymptotically Minkowskian* spacetimes essential self-adjointness was recently proven by Vasy and Nakamura-Taira. Essential self-adjointness is destroyed by (space-like or time-like) *boundaries*—this can be sometimes repaired by boundary conditions.

There exists also a different definition of Feynman propagators based on a *time-ordered expectation* of quantum fields in a state. Let $\hat{\psi}(x)$ be the quantum field satisfying

$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x,y).$$

Let Ω_α be any *Fock vacuum* (in other words, *pure quasifree state*). Set

$$G_\alpha^{(+)} = (\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\alpha), \quad G_\alpha^{(-)} = (\Omega_\alpha|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\alpha),$$

$$-\mathrm{i}G_\alpha^{\mathrm{F}} = (\Omega_\alpha|\mathrm{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\alpha), \quad \mathrm{i}G_\alpha^{\overline{\mathrm{F}}} = (\Omega_\alpha|\overline{\mathrm{T}}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\alpha).$$

We have $$G_\alpha^{\mathrm{F}}(x,y) + G_\alpha^{\overline{\mathrm{F}}}(x,y) = G^{\vee}(x,y) + G^{\wedge}(x,y).$$

We say that Ω_α is *Hadamard* if the singularities of $G_\alpha^{(+)}$ are similar to those on the Minkowski space.

It useful to extend the above definitions of 2-point functions, Feynman and antiFeynman propagators to *pairs of vacua* Ω_α and Ω_β:

$$G^{(+)}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\hat{\psi}(x)\hat{\psi}^*(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$G^{(-)}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\hat{\psi}^*(x)\hat{\psi}(y)\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$-\mathrm{i}G^{\mathrm{F}}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\mathrm{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)},$$

$$\mathrm{i}G^{\overline{\mathrm{F}}}_{\alpha\beta}(x,y) = \frac{(\Omega_\alpha|\overline{\mathrm{T}}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega_\beta)}{(\Omega_\alpha|\Omega_\beta)}.$$

Note that they satisfy

$$
\begin{aligned}
(-\Box_x + m^2) G^{(+)/(-)}_{\alpha\beta}(x,y) &= 0, \\
(-\Box_x + m^2) G^{\mathrm{F}/\overline{\mathrm{F}}}_{\alpha\beta}(x,y) &= \delta(x,y), \\
\mathrm{i}(G^{(+)}_{\alpha\beta} - G^{(-)}_{\alpha\beta}) &= G^{\mathrm{PJ}} = G^{\vee} - G^{\wedge}, \\
G^{\mathrm{F}}_{\alpha\beta} + G^{\overline{\mathrm{F}}}_{\alpha\beta} &= G^{\vee} + G^{\wedge}, \\
G^{(+)}_{\alpha\beta}(x,y) = \overline{G^{(-)}_{\beta\alpha}(y,x)}, &\qquad G^{\mathrm{F}}_{\alpha\beta}(x,y) = \overline{G^{\overline{\mathrm{F}}}_{\beta\alpha}(y,x)}.
\end{aligned}
$$

$G^{(+)}_{\alpha\beta}$, $G^{(-)}_{\beta\alpha}$, $G^{\mathrm{F}}_{\alpha\beta}$, $G^{\overline{\mathrm{F}}}_{\beta\alpha}$ can be defined in a purely operator-theoretic way, using the Krein space of Cauchy data—then we neither have to divide by zero, nor invoke QFT!

Suppose that the Klein-Gordon equation is *stationary* (does not depend on time) and *stable* (the classical Hamiltonian is positive). Then there is a distinguished vacuum Ω, given by the space of *positive frequency modes* of the generator of dynamics. It is then easy to show (D. Siemssen and JD) that $-\Box + m^2$ is essentially self-adjoint and the operator-theoretic Feynman propagator corresponds to Ω:

$$-\mathrm{i}G^{\mathrm{F}}_{\mathrm{op}} = (\Omega|\mathrm{T}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega),$$

$$\mathrm{i}G^{\overline{\mathrm{F}}}_{\mathrm{op}} = (\Omega|\overline{\mathrm{T}}(\hat{\psi}(x)\hat{\psi}^*(y))\Omega).$$

If M is *asymptotically stationary and stable in the future and past* then we have two natural states: the *in-vacuum* Ω_- and the *out-vacuum* Ω_+. As proven by Gérard and Wrochna, they are Hadamard

Introduce the *out-in Feynman propagator* and the *in-out antiFeynman propagator*

$$-\mathrm{i}G^{\mathrm{F}}_{+-}(x,y) = \frac{\big(\Omega_+|\mathrm{T}\big(\hat\psi(x)\hat\psi^*(y)\big)\Omega_-\big)}{\big(\Omega_+|\Omega_-\big)},$$

$$\mathrm{i}G^{\overline{\mathrm{F}}}_{-+}(x,y) = \frac{\big(\Omega_-|\overline{\mathrm{T}}\big(\hat\psi(x)\hat\psi^*(y)\big)\Omega_+\big)}{\big(\Omega_-|\Omega_+\big)}.$$

By the *Wick Theorem*, they appear in the evaluation of *Feynman diagrams* for the scattering operator resp. its inverse.

As proven by (D.Siemssen and JD) in great generality, (and also by Gérard and Wrochna for asymptotically Minkowski spacetimes) G^{F}_{+-} and $G^{\overline{\mathrm{F}}}_{-+}$ are well defined.

One can heuristically derive, and under some technical assumptions prove rigorously (Vasy and Nakamura–Taira) that they coincide with the operator-theoretic propagators:

$$\begin{aligned} G^{\mathrm{F}}_{\mathrm{op}} &= G^{\mathrm{F}}_{+-}, \\ G^{\overline{\mathrm{F}}}_{\mathrm{op}} &= G^{\overline{\mathrm{F}}}_{-+}. \end{aligned}$$

Assume now that M is *globally hyperbolic* and $-\Box$ *is essentially self-adjoint*. (If not, choose a self-adjoint extension).

We will say that $-\Box + m^2$ is *special* if

$$\operatorname{supp}\left(G^{\mathrm{F}}_{\mathrm{op}}(\cdot, y) + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(\cdot, y)\right) \subset \text{causal shadow of}\{y\}.$$

Then we by splitting the above distribution into the future and past lightcones we expect the identity involving the forward and backward propagators:

$$G^{\mathrm{F}}_{\mathrm{op}}(x, y) + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x, y) = G^{\vee}(x, y) + G^{\wedge}(x, y).$$

Special Klein-Gordon equations are *superconvenient*! There exist good techniques to compute the Feynman and antiFeynman propagators (because they are defined in the framework of operator theory). The forward/backward propagators can then be computed as

$$G^{\vee/\wedge}(x,y) := \theta(\pm x^0 \mp y^0)\big(G^{\mathrm{F}}_{\mathrm{op}}(x,y) + G^{\overline{\mathrm{F}}}_{\mathrm{op}}(x,y)\big).$$

As usual, we then set $G^{\mathrm{PJ}} := G^{\vee} - G^{\wedge}$. More interestingly, we have a natural candidate for the *two-point function of a distinguished state*:

$$(\Omega\,|\,\hat{\psi}(x)\hat{\psi}^*(y)\Omega) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}}_{\mathrm{op}} - G^{\wedge}) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}}_{\mathrm{op}} + G^{\vee}),$$

$$(\Omega\,|\,\hat{\psi}^*(x)\hat{\psi}(y)\Omega) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}}_{\mathrm{op}} + G^{\wedge}) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}}_{\mathrm{op}} - G^{\vee}).$$

Recall that for any state α

$$G_\alpha^{\mathrm{F}}(x,y) + G_\alpha^{\overline{\mathrm{F}}}(x,y) = G^\vee(x,y) + G^\wedge(x,y).$$

Hence if

$$\Omega_- = \Omega_+,$$

then $-\Box + m^2$ is special.

This is in particular true if M is stationary and stable—hence they are special.

III. EXAMPLES OF SPACETIMES AND THEIR PROPAGATORS

Stationary Klein-Gordon equations are especially easy, as we discussed above. This includes the Minkowski space. They are special if they are *stable*, that is the Hamiltonian is positive definite (which corresponds to $m^2 \geq 0$).

For *tachyonic* stationary Klein-Gordon equations, that is with $m^2 < 0$, we can also define all four Green's functions. However they are not special! (And, of course, we do not have a physical state).

Consider a $1+0$ *dimensional spacetime*. In view of applications to FLRW spacetimes, assume that it is perturbed by a time-dependent potential. Thus the Klein-Gordon operator has the form of a *1-dimensional Schrödinger operator*

$$K = -H + m^2, \quad H := -\partial_t^2 + V(t).$$

It is special if H is *reflectionless* at the energy m^2.

For instance, the *symmetric Scarf Hamiltonian*

$$-\partial_t^2 - \frac{\alpha^2 - \frac{1}{4}}{\cosh^2 t}$$

is reflectionless at all energies for $\alpha \in \mathbb{Z} + \frac{1}{2}$.

The *deSitter space* is defined as the submanifold of the $d+1$-dimensional Minkowski *ambient space*:

$$\mathrm{dS}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}.$$

One can look for the Feynman propagator by solving the equation

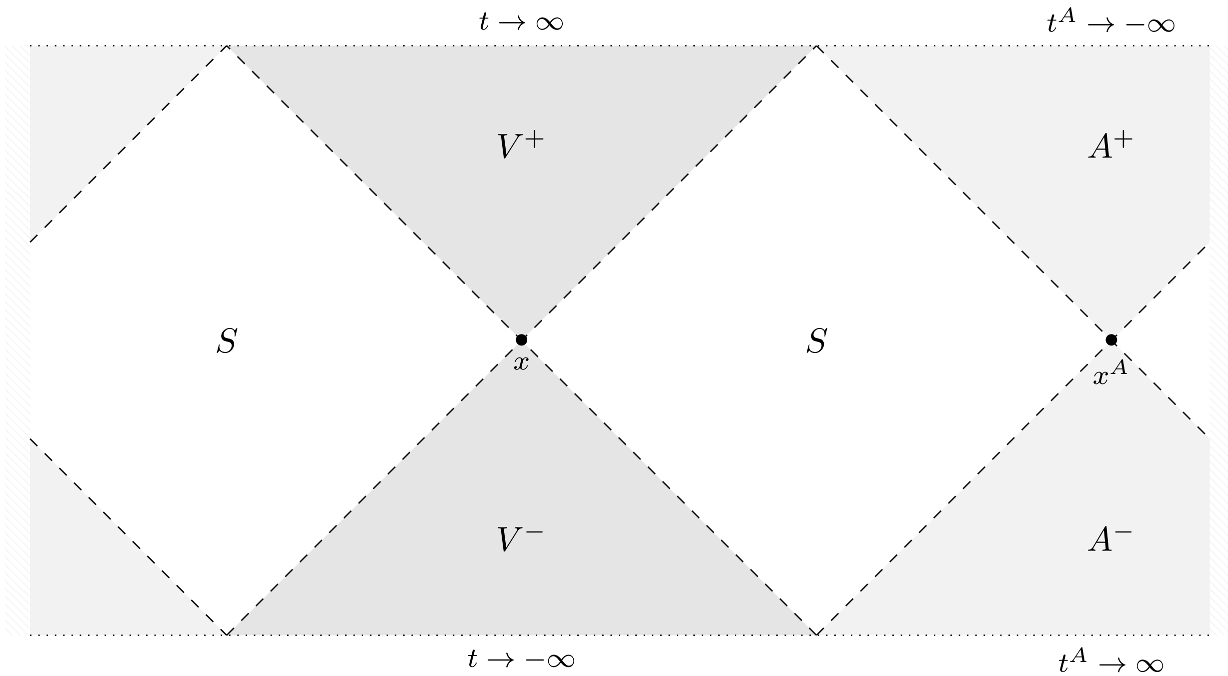
$$(-\Box_x + m^2)G^{\mathrm{F}}(x,y) = \delta(x-y),$$

and requiring that $G^{\mathrm{F}}(x,y) = G^{\mathrm{F}}(w)$, where $w = x{\cdot}y$ is the product of the vectors in the ambient space. We obtain the *Gegenbauer equation*

$$\Big((1-w^2)\partial_w^2 - dw\partial_w - (\tfrac{d-1}{2})^2 + m^2\Big)G^{\mathrm{F}}(w) = 0.$$

We demand the singularities of G^{F} are similar to those of the Feynman propagator on the Minkowski space.

Figure 2: $V^{\pm} := \{Z(x,x') > 1 \mid t(x,x') \gtrless 0\}$, $A^{\pm} := \{Z(x,x') < -1 \mid t(x^A,x') \lessgtr 0\}$ and $S := \{|Z(x,x')| < 1\}$.



Assuming $m > \frac{d-1}{2}$ and setting $\nu := \sqrt{m^2 - (\frac{d-1}{2})^2}$ we obtain

$$G_{\rm E}^{{\rm F}/\overline{\rm F}}(w) = \pm {\rm i}\frac{\Gamma(\frac{d-1}{2}+{\rm i}\nu)\Gamma(\frac{d-1}{2}-{\rm i}\nu)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,{\rm i}\nu}\big(-w \pm {\rm i}0\big).$$

Above, $\mathbf{S}_{\alpha,\nu}$ is the *Gegenbauer function* regular at 1 and equal $\frac{1}{\Gamma(\alpha+1)}$ there. It satisfies

$$G_{\rm E}^{\rm F} + G_{\rm E}^{\overline{\rm F}} = G^{\vee} + G^{\wedge}.$$

We can compute forward/backward propagators, and the distinguished two-point function, called the *Euclidean state* (because it is obtained by the Wick rotation from the Euclidean sphere). It is the unique deSitter invariant Hadamard state.

The d'Alembertian on $C_{\rm c}^\infty({\rm dS}^d)$ is essentialy self-adjoint and thus one can define the operator-theoretic Feynman and antiFeynman propagator. However, it is different from the Euclidean one:

$$G_{\rm E}^{\rm F} \neq G_{\rm op}^{\rm F}, \qquad G_{\rm E}^{\overline{\rm F}} \neq G_{\rm op}^{\overline{\rm F}}.$$

Note that the deSitter space is quite pathological—in particular it is not asymptotically stationary, and the Euclidean state is neither the *in-state* nor the *out-state*.

There exists a family of deSitter invariant states parametrized by a complex parameter, called *alpha-vacua*. Among them there is the Euclidean state, an in-state and an out-state. The out-in Feynman propagators coincides with the operator-theoretic Feynman propagators and is given by

$$G^{\mathrm{F}}_{+-}(w) = \frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\nu)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\sinh\pi\nu}\Big(\mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-w-\mathrm{i}0\big)-\mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-w+\mathrm{i}0\big)\Big), \quad \text{odd } d;$$

$$G^{\mathrm{F}}_{+-}(w) = \frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\nu)}{2^{2+\mathrm{i}\nu}(2\pi)^{\frac{d-1}{2}}\cosh\pi\nu}\Big(\mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-w-\mathrm{i}0\big)+\mathbf{Z}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-w+\mathrm{i}0\big)\Big), \quad \text{even } d.$$

where $\mathbf{Z}_{\alpha,\lambda}$ is the *Gegenbauer function* behaving as $\frac{w^{-\frac{1}{2}-\alpha-\lambda}}{\Gamma(\lambda+1)}$ at $w\to+\infty$.

In *odd dimensions* and with $m^2 > (\frac{d-1}{2})^2$, the deSitter space is special and the *out and in vacua coincide*. This is not the case in even dimensions!

There is an alternative approach to the deSitter space based on global coordinates

$$X_0 = \sinh t, \quad X_i = \cosh t \hat{x}_i, \quad \hat{x} \in \mathbb{S}^{d-1}$$

yielding the metric $-\,\mathrm{d}t^2 + \cosh^2 t\,\mathrm{d}\Omega^2$. This has a FLRW form and yields the Schrödinger operator

$$-\partial_t^2 - \frac{\left(\frac{d-2}{2}\right)^2 - \frac{1}{4} - \Delta_{\mathbb{S}^{d-1}}}{\cosh^2 t} + \left(\frac{d-1}{2}\right)^2.$$

The spectrum of $-\Delta_{\mathbb{S}^{d-1}}$ is $\{l(l+d-2)\ :\ l = 0, 1, 2, \dots\}$, hence we obtain the symmetric Scarf potential with $\alpha = \frac{d-2}{2} + l$. Thus all modes are reflectionless iff d is odd. Consequently, *all modes are special iff d is odd, and they are not if d is even*.

The *Anti-deSitter space* is defined as

$$\mathrm{AdS}^d := \{(X,Y)\in\mathbb{R}^2\times\mathbb{R}^{d-1}\ :\ -X_1^2-X_2^2+Y_1^2+\cdots+Y_{d-1}^2=-1\}.$$

It is stationary, however has timelike loops. Introduce the coordinates

$$X_1=\frac{\cos t}{\cos\rho},\quad X_2=\frac{\sin t}{\cos\rho},\quad Y_i=\tan\rho\hat{y}_i;$$

with the metric $$\frac{1}{\cos^2\rho}(-\mathrm{d}t^2+\mathrm{d}\rho^2+\sin^2\rho\,\mathrm{d}\Omega^2).$$

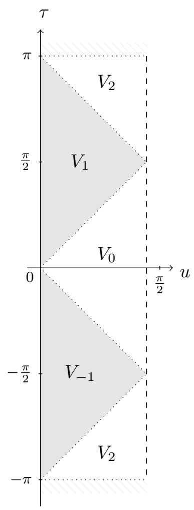
where $t\in]-\pi,\pi]$. By taking the *universal covering* of the Anti-deSitter space we remove timelike loops. In coordinates this means $t\in\mathbb{R}$,

The d'Alembertian is essentially self-adjoint. We again set $w:=x\cdot y$ from the ambient space. For $m^2>-(\frac{d-1}{2})^2$, with $\nu:=\sqrt{(\frac{d-1}{2})^2+m^2}$, we obtain on V_n

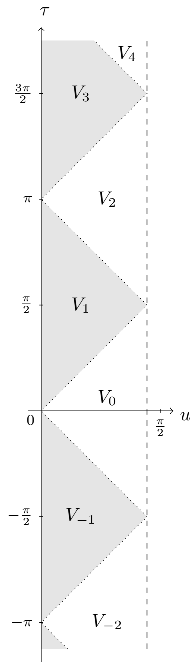
$$G_{\mathrm{op}}^{\mathrm{F}/\overline{\mathrm{F}}}(x,x')=\pm\,\mathrm{i}\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2}+\nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^{\nu}}\mathrm{e}^{\mp\mathrm{i}|n|(\frac{d-1}{2}+\nu)\pi}\mathbf{Z}_{\frac{d}{2}-1,\nu}\Big(-(-1)^n w\pm(-1)^n\mathrm{i}0s\Big),$$

where $s=0$ on V_{2n} and $s=(-1)^n$ on $V_{2n-1}\cup V_{-2n+1}$.

Figure 3: (a) Proper anti-deSitter space, (b) Universal cover of anti-deSitter space.



(a)



(b)

In the following, it will be useful to know some properties of the *trigonometric Pöschl-Teller Hamiltonian*:

$$H := -\partial_\rho^2 + \frac{\alpha^2 - \frac{1}{4}}{\sin^2 \rho} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 \rho}.$$

This Hamiltonian, as an operator on $L^2[0, \frac{\pi}{2}]$, is essentially self-adjoint iff $\alpha^2 \geq 1$ and $\beta^2 \geq 1$, and has a positive Friedrichs extension if $\alpha^2 \geq 0$ and $\beta^2 \geq 0$. If $\alpha^2 < 0$ or $\beta^2 < 0$, then all its extensions are unbounded from below.

The Anti-deSitter space, even after taking its universal covering, is still not globally hyperbolic: it has trajectories that *escape to infinity in finite time*.

Consider now the Klein-Gordon operator on Anti-deSitter:

$$
\begin{aligned}
&(\tan\rho)^{\frac{d-2}{2}}(-\Box+m^2)(\tan\rho)^{-\frac{d-2}{2}} \\
&=\cos^2\rho\Big(\partial_t^2-\partial_\rho^2+\frac{\big(\frac{d-3}{2}\big)^2-\frac14-\Delta_{\mathbb{S}^{d-2}}}{\sin^2\rho}+\frac{\big(\frac{d-1}{2}\big)^2-\frac14+m^2}{\cos^2\rho}\Big) \\
&=\cos^2\rho\big(\partial_t^2+H\big),
\end{aligned}
$$

where H is the trigonometric Pöschl-Teller Hamiltonian. $\rho=0$ is a coordinate singularity. $\rho=\frac{\pi}{2}$ is the spatial infinity, where classical particles may escape. Following Wald-Ishibashi, we note that H is self-adjoint for $m^2\geq 1-(\frac{d-1}{2})^2$. For $m^2\geq -(\frac{d-1}{2})^2$, we need to take the Friedrichs extension of H. In all these cases the *Anti-deSitter space is special!* Only for $m^2< -(\frac{d-1}{2})^2$ we do not have distinguished forward and backward propagators (and of course the specialty breaks down).

Thank you for your attention