The Matrix Toda Equation

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Overview

Background. One-sided Toda lattice system

A one-sided Toda lattice system is a system of particles on the real line with coordinates $\{x_n\}_{n=1}^{\infty}$:

$$\ddot{x}_n = e^{-(x_n - x_{n-1})} - e^{-(x_{n+1} - x_n)}, \quad n = 1, 2, \ldots \tag{1}$$

(1) is equivalent to

$$\begin{cases} \dot{x}_n = y_n, \\ \dot{y}_n = e^{-(x_n - x_{n-1})} - e^{-(x_{n+1} - x_n)}. \end{cases} \tag{2}$$

With Flaschka transformation

$$\lambda_{n+1} = e^{-(x_n - x_{n-1})} - e^{-(x_{n+1} - x_n)}, \quad \alpha_n = -\dot{x}_n, \tag{3}$$

then (2) is equivalent to

$$\begin{cases} \dot{\alpha}_n = \lambda_{n+1} - \lambda_n, \\ \dot{\lambda}_{n+1} = \lambda_{n+1}(\alpha_{n+1} - \alpha_n), \quad n = 1, 2, \ldots. \end{cases} \tag{4}$$

- F. Peherstorfer, On Toda and orthogonal polynomials, J. Comput. Appl. Math. **133**, 2001.

Background. Hurwitz polynomial

A polynomial f_n is called a Hurwitz polynomial if all its roots are located in the open left-hand side of the complex plane.

Theorem 16. Chapter 15. F.R. Gantmacher, Theory of matrices

A real polynomial of degree n $f(z) = h(z^2) + z\,g(z^2)$ is a Hurwitz polynomial if and only if

$$\frac{g(z)}{h(z)} = c_0 + \cfrac{1}{d_0 z + \cfrac{1}{c_1 + \cfrac{1}{\ddots + c_{m-2} + \cfrac{1}{d_{m-1}u + c_m^{-1}}}}}, \tag{5}$$

holds with nonnegative c_0 and positive $c_1, \ldots, c_m$, $d_0, \ldots, d_{m-1}$. Here $c_0 > 0$ when n is odd and $c_0 = 0$ when n is even.

Motivation

Our motivation is to attain applications

- to the control theory of systems described by block differential equations and
- to robust control: Systems with uncertain physical parameters in the matrix case.

- J. Ackermann, A. Bartlett, D. Kaesbauer, W. Sienel, R. Steinhauser, Robust Control, Springer: Berlin, 1993.
- V.L. Kharitonov, Asymptotic stability of an equilibrium position of a family of systems of linear differential equations, Differential Equations 14, 1979.

- ACR, Hurwitz polynomials and orthogonal polynomials generated by Routh-Markov parameters, Mediterranean Journal of Mathematics, 15:40, 2018.
- ACR, The Kharitonov theorem and robust stabilization via orthogonal polynomials, to appear in Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh., 86, 2017.

- ACR, Extended set of solutions of a bounded finite-time stabilization problem via the controllability function, IMA Journal of Mathematical Control and Information, 38(4), 2021.
- ACR; Korobov's Controllability function as motion time: Extension of the solution set of the synthesis problem, Journal of Mathematical Physics, Analysis, Geometry, 19(3), 2023.

The matrix moment problem

Let $(s_n)_{n\geq 0}$ be a sequence of $q \times q$ Hermitian matrices. Find the set of positive $q \times q$ measures σ on I such that

$$s_n = \int_I t^n d\sigma(t), \quad n \geq 0 \tag{6}$$

- $I = [0, \infty)$, Stieltjes moment problem,
- $I = (-\infty, \infty)$, Hamburger moment problem,
- $I = [a, b]$, Hausdorff moment problem.

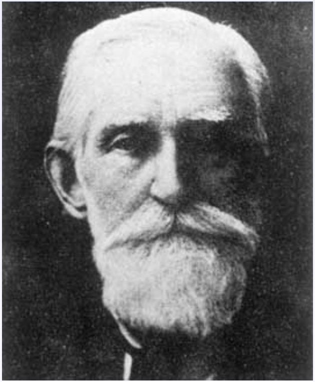
- Existence of the solution of the matrix moment problem (MMP).
- Determinateness of the matrix moment problem.
- Description of the set of solutions.

If the solution is unique the MMP is called determined.
In the other case the MMP is caled indetermined.

M.G. Krein, *Infinite J-matrices and a matrix moment problem*(Russian), Dokl. Akad. Nauk SSSR 69, 1949.

P.L. Chebyschev (1821-1894)

T.J. Stieltjes (1856-1894)



$\int_a^b u^k f(u) du$

$\int_0^\infty u^k d\sigma(u)$

Stieltjes, T.J.: *Recherches sur les fractions continues*, Annales Fac. Toulouse, **8**, 1–22, 1894, 9, 1–47, 1895,

Čebyšev, P.L.: *Polnoye sobranie sochinenii*, volume **2**, Matematicheski Analis, Izd. AN SSSR, Moscow, 1947

N.I. Akhiezer (1901-1980)

M.G.Krein (1907-1989)

Akhiezer N.I., Krein M.G.: *Ueber Fouriersche Reihen beschraenkter summierbarer Funktionen und ein neus Extremumproblem II*, Comm. de la Soc.Math. de Kharkoff, sér. 4, T.10, 1934.

Akhiezer, N.I.: *The Classical Moment Problem*, Oliver and Boyd, London, 1965.

Akhiezer N.I., Krein M.G.: Translations of Mathematical Monographs, Vol. **2**, Amer. Math. Soc., Providence, R.I. (1938)1962.

Krein, M.G.; Nudelman A.A.: *The Markov Moment Problem and Extremal Problems*. Translation Math. Monographs, Vol. 50, AMS, 1977.

Vladimir Petrovich Potapov,
Odessa (1914)–Kharkov (1980)

V.P. Potapov, *The multiplicative structure of J-nonexpansive matrix functions*, Trudy Moskov. Mat. Ob. 4, 1955.
A.V. Efimov, V.P. Potapov, *J–expansive matrix-valued functions and their role in the analytical theory of electrical circuits*, Russian Math. Surveys, 28, 1973.

Irina Vasilevna Kovalishina,
– 2007, Kharkov

I. V. Kovalishina, *Analytic theory of a class of interpolation problems* Izv. Akad. Nauk SSSR Ser. Mat., Volume 47, Issue 3, 1983.

Existence of solutions

The (Hamburger) Stieltjes MMP has a solution iff ($H_{1,n} \geq 0$) and $H_{2,n} \geq 0$, $n = 0, 1, \ldots$ where

$$H_{1,n} := \begin{pmatrix} s_0 & s_1 & \ldots & s_j \\ s_1 & s_2 & \ldots & \ldots \\ \vdots & \ldots & \ldots & \ldots \\ s_n & s_{n+1} & \ldots & s_{2n} \end{pmatrix}, \quad H_{2,n} := \begin{pmatrix} s_1 & s_2 & \ldots & s_{n+1} \\ s_2 & s_3 & \ldots & \ldots \\ \vdots & \ldots & \ldots & \ldots \\ s_{n+1} & s_{n+2} & \ldots & s_{2n+1} \end{pmatrix}.$$

The **truncated** Stieltjes MMP with finite number of moments $(s_j)_{j=0}^{2n}$ or $(s_j)_{j=0}^{2n+1}$ with $H_{1,n} > 0$, $H_{2,n-1} > 0$ was solved by Yury Dyukarev in 1981 and 1999, respectively.

Yu.M. Dyukarev, *The Stieltjes matrix moment problem*, VINITI (Moscow) 1981.

Yu.M. Dyukarev, *A general scheme for solving interpolation problems in the Stieltjes class that is based on consistent representations of pairs of nonnegative operators. I.* (Russian) Mat. Fiz. Anal. Geom. 6, 1999.

The **truncated** Hamburger MMP with $(s_j)_{j=0}^{2n}$ moments and $H_{1,n} > 0$ was studied by:

I.V. Kovalishina, *Analytic theory of a class of interpolation problems* Izv. Akad. Nauk SSSR Ser. Mat.47(3), 1983.

$q \times q$ positive matrix measure $\sigma \in \mathcal{M}(\mathbb{R}, (s_j)_{j=0}^{2n})$ associated with an analytic function $s \in \mathcal{L}(\mathbb{R}, (s_j)_{j=0}^{2n})$

Truncated Hamburger MMP: Instead of σ one looks for the associated $s(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\sigma(t)$, s is holomorphic in $\mathbb{C} \setminus \mathbb{R}$ and $\mathrm{Im}\ s \geq 0$, $\mathrm{Im}\ z > 0$;

$$\lim_{z \to \infty} z^{2j+1} \left(s(z) + \frac{s_0}{z} + \frac{s_1}{z^2} + \ldots + \frac{s_{2j-1}}{z^{2j}} \right) = -s_{2j},$$

for $\delta \leq \arg z \leq \pi - \delta$, $0 < \delta < \pi/2$.

Stieltjes–Perron inverse formula:

$$\sigma(t_2) - \sigma(t_1) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{t_1}^{t_2} \mathrm{Im}\ s(x + i\epsilon) dx, \quad \sigma(t) = \frac{\sigma(t+0) + \sigma(t-0)}{2}.$$

Example of a 2×2 positive measure on $\mathbb{R}$:

$$\sigma(t) = \begin{cases} 0_{2 \times 2}, & t < \frac{1}{2} \\ \begin{pmatrix} 1 & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 1 \end{pmatrix} & t \geq \frac{1}{2} \end{cases}$$

Definition 1 (V.P. Potapov's Fundamental Matrix Inequality (FMI))

The function s is called a solution of the FMI if: i) s is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. ii) s satisfies:

$$\left(\begin{array}{c|c} H_j & R_{T_j}(z)\,[\,v_j s(z) - u_j\,] \\ \hline (R_{T_j}(z)\,[\,v_j s(z) - u_j\,])^* & (s(z) - s^*(z))/(z - \bar{z}) \end{array}\right) \geq 0 \qquad (FMI)$$

The FMI for 3 moments $s_0,\ s_1,\ s_2$ is the following:

$$\left(\begin{array}{cc|c} s_0 & s_1 & s(z) \\ s_1 & s_2 & zs(z) - s_0 \\ \hline s^*(z) & \bar{z}s^*(z) - s_0^* & \dfrac{s(z) - s^*(z)}{z - \bar{z}} \end{array}\right) \geq 0. \qquad (7)$$

Theorem 1 (V. Potapov-I. Kovalishina)

s is the solution of the FMI iff s is the associated solution of the truncated Hamburger MMP.

Potapov's approach. Non degenerate case, $H_j > 0$.

$$\left(\begin{array}{c|c} H_j & R_{T_j}(z)\,[\,v_j s(z) - u_j\,] \\ \hline (R_{T_j}(z)\,[\,v_j s(z) - u_j\,])^* & (s(z) - s^*(z))/(z - \bar{z}) \end{array}\right) \geq 0 \qquad (FMI)$$

$$H_j = \begin{pmatrix} s_0 & s_1 & \dots & s_j \\ s_1 & s_2 & \dots & \dots \\ \vdots & & & \\ & \dots & \dots & \dots \\ s_j & s_{j+1} & \dots & s_{2j} \end{pmatrix} u_j = \begin{pmatrix} 0 \\ -s_0 \\ \vdots \\ -s_{j-1} \end{pmatrix}, v_j = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, R_j(z) = \begin{pmatrix} I & \dots & 0 & 0 \\ zI & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ z^j I & \dots & zI & I \end{pmatrix},$$

Resolvent matrix for the truncated Hamburger MMP (THMP), $H_j > 0$.

$$U_j(z) := \left(\begin{array}{c|c} I + zu_j^*[R_j(\bar{z})]^* H_j^{-1} v_j & -zu_j^*[R_j(\bar{z})]^* H_j^{-1} v_j \\ \hline zv_j^*[R_j(\bar{z})]^* H_j^{-1} v_j) & I + zv_j^*[R_j(\bar{z})]^* H_j^{-1} u_j \end{array}\right).$$

$$J := \begin{pmatrix} 0 & -i\,I \\ i\,I & 0 \end{pmatrix}, \qquad \begin{pmatrix} s(z) \\ I \end{pmatrix}^* \frac{U^{-1^*}(z) J U^{-1}(z)}{i(\bar{z} - z)} \begin{pmatrix} s(z) \\ I \end{pmatrix} \geq 0. \qquad (FMI2)$$

$$\text{THMP solutions: } s(z) = \frac{U_{11,j}(z) + U_{12,j}(z)\,\omega}{U_{21,j}(z) + U_{22,j}(z)\,\omega}, \quad \omega(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}, \quad \begin{pmatrix} I \\ \omega \end{pmatrix}.$$

For A, $B \in \mathbb{C}^{q \times q}$ with B invertible, set $\frac{A}{B} := AB^{-1}$.

Signature matrix. $J := \begin{pmatrix} 0 & -i\, I_q \\ i\, I_q & 0 \end{pmatrix}$, $J^2 = I_{2q}$, $J^* = J$

Definition 2

A is J–contractive (J–expansive) if $J - A^* J A \geq 0$ ($A^* J A - J \geq 0$).

$$U_j(z) := I_{2q} - i\, z \begin{pmatrix} u_j \\ v_j \end{pmatrix}^* R_j^*(\bar{z}) H_j^{-1} (u_j,\, v_j) J$$

$$J - U(z) J U^*(z) = \begin{cases} \geq 0 & \mathrm{Im}\, z < 0 \\ = 0 & \mathrm{Im}\, z = 0 \\ \leq 0 & \mathrm{Im}\, z > 0 \end{cases}$$

$$U^{-1}(z) = J\, U^*(\bar{z})\, J.$$

Fundamental identity:

$$T_j\, H_j - H_j\, T_j^* = v_j\, u_j^* - u_j\, v_j^*, \quad T_j = \begin{pmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \end{pmatrix}$$

Dyukarev's resolvent matrix of the Stieltjes MMP

$$\alpha^{(2j)}(z) := I_q - z u_{1,j}^* R_j^*(\bar z) H_{1,j}^{-1} v_j, \qquad \beta^{(2j-1)}(z) := u_{2,j-1}^* R_{j-1}^*(\bar z) H_{2,j-1}^{-1} u_{2,j-1},$$

$$\gamma^{(2j)}(z) := -z v_j^* R_j^*(\bar z) H_{1,j}^{-1} v_{1,j}, \qquad \delta^{(2j-1)}(z) := I_q + z v_{j-1}^* R_{j-1}^*(\bar z) H_{2,j-1}^{-1} u_{2,j-1},$$

$$U^{(2j)}(z) := \begin{pmatrix} \alpha^{(2j)}(z) & \beta^{(2j-1)}(z) \\ \gamma^{(2j)}(z) & \delta^{(2j-1)}(z) \end{pmatrix} \qquad U^{(2j+1)} := \begin{pmatrix} \alpha^{(2j)}(z) & \beta^{(2j)}(z) \\ \gamma^{(2j)}(z) & \delta^{(2j)}(z) \end{pmatrix} \tag{8}$$

$$U^{(2j)}(z) = \begin{pmatrix} I_q & 0_q \\ -zM_0 & I_q \end{pmatrix} \begin{pmatrix} I_q & L_0 \\ 0_q & I_q \end{pmatrix} \cdots \begin{pmatrix} I_q & L_{j-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -zM_j & I_q \end{pmatrix}, \tag{9}$$

$$U^{(2j+1)}(z) = \begin{pmatrix} I_q & 0_q \\ -zM_0 & I_q \end{pmatrix} \begin{pmatrix} I_q & L_0 \\ 0_q & I_q \end{pmatrix} \cdots \begin{pmatrix} I_q & 0_q \\ -zM_j & I_q \end{pmatrix} \begin{pmatrix} I_q & L_j \\ 0_q & I_q \end{pmatrix}, \tag{10}$$

Matrix case.

$$M_0 := s_0^{-1} > 0, \quad L_0 = s_0 s_1^{-1} s_0 > 0,$$

$$M_j := v_j^* H_{1,j}^{-1} v_j - v_{j-1}^* H_{1,j-1}^{-1} v_{j-1} > 0,$$

$$L_j := u_{2,j}^* H_{2,j}^{-1} u_{2,j} - u_{2,j-1}^* H_{2,j-1}^{-1} u_{2,j-1} > 0.$$

Scalar case. Gantmacher F/Krein M (1950)

$$l_j := \frac{\Delta_j^2}{\Delta_j^{(1)} \Delta_{j-1}^{(1)}}, \; m_j := \frac{[\Delta_{j-1}^{(1)}]^2}{\Delta_j \Delta_{j-1}}, j \geq 0$$

$$\Delta_j := \det(s_{i+k})_{i,k=0}^j, \; \Delta_j^{(1)} := \det(s_{i+k+1})_{i,k=0}^j,$$

$$\Delta_{-1} := \Delta_{-1}^{(1)} = 1.$$

Yu.M. Dyukarev, *Indeterminacy criteria for the Stieltjes matrix moment problem*, Mathematical Notes 75 (2004).

Yu.M. Dyukarev, *Examples of block Jacobi matrices that generate symmetric operators with arbitrary possible deficiency numbers.* Sb. Math. 201, 2010.

Orthogonal Matrix Polynomials (OMP)

Let σ be a $q \times q$ positive measure on $[0, +\infty)$. We say that the matrix polynomial $P(t) = \sum_{k=0}^{n} A_k t^k$ with $q \times q$ matrix coefficients A_k has degree n if A_n is non-zero. A finite sequence of matrix polynomials $(P_j)_{j=0}^{n}$ is called Orthogonal Matrix Polynomials (OMP) on $[0, \infty)$ with respect to σ if:

i) $\deg P_j(t) = j$ for all $j \in \{0, \ldots, n\}$.

ii) $\int_{[0,+\infty)} P_m(t) d\sigma(t) P_j^*(t) = \delta_{mj} C_{mj}$, for all $m, j \in \{0, \ldots, n\}$ where C_{mj} is a constant non zero $q \times q$ matrix.

If C_{mj} is the identity matrix I_q, the sequence of matrix polynomials $\{P_j\}_{j=0}^{n}$ is called Stieltjes Orthonormal Matrix Polynomials.

D. Damanik, A. Pushnitski and B. Simon, *The analytic theory of matrix orthogonal polynomials*, Surv. Approx. Theory 4, 2008.

A.J. Durán, P. López–Rodríguez, 2007; H. Dette, B. Reuther, W.J. Studden, M. Zygmunt, 2006 F.A. Grünbaum, I. Pacharoni, J.A. Tirao, 2003.

Let $(s_j)_{j=0}^{2n}$ $((s_j)_{j=0}^{2n+1})$ be a positive sequence, i.e., $H_{1,n} > 0$ and $H_{2,n-1} > 0$ $(H_{2,n} > 0)$.

$$P_{1,0}(z) := I_q,\ Q_{1,0}(z) := 0_q,\ P_{2,0}(z) := I_q,\ Q_{2,0}(z) := s_0,$$

$$P_{1,j}(z) := (-Y_{1,j}^* H_{1,j-1}^{-1},\ I_q) R_j(z) v_j, \quad P_{2,j}(z) := (-Y_{2,j}^* H_{2,j-1}^{-1},\ I_q) R_j(z) v_j, \tag{11}$$

$$Q_{1,j}(z) := -(-Y_{1,j}^* H_{1,j-1}^{-1},\ I_q) R_j(z) u_{1,j}, \quad Q_{2,j}(z) := -(-Y_{2,j}^* H_{2,j-1}^{-1},\ I_q) R_j(z) u_{2,j}. \tag{12}$$

$$u_{1j} = \begin{pmatrix} 0 \\ -s_0 \\ \vdots \\ -s_{j-1} \end{pmatrix},\ u_{2j} = \begin{pmatrix} -s_0 \\ -s_1 \\ \vdots \\ -s_j \end{pmatrix},\ v_j = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ R_j(z) = \begin{pmatrix} I & \dots & 0 & 0 \\ zI & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ z^j I & \dots & zI & I \end{pmatrix},$$

$$Y_{1,j} := \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_{2j-1} \end{pmatrix},\ 1 \le j \le n, \qquad Y_{2,j} := \begin{pmatrix} s_{j+1} \\ s_j \\ \vdots \\ s_{2j} \end{pmatrix},\ 2 \le j \le n,\ \ \widehat{H}_{1,0} := s_0,\ \ \widehat{H}_{2,0} := s_1.$$

$$\widehat{H}_{1,j} := s_{2j} - Y_{1,j}^* H_{1,j-1}^{-1} Y_{1,j}, \quad \widehat{H}_{2,j} := s_{2j+1} - Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j}, \quad j \ge 1. \tag{13}$$

Proposition 1

The polynomials $P_{1,j}$ and $P_{2,j}$ are OMPs with respect to σ and $t\sigma$, respectively. More precisely,

$$\int_{[0,+\infty)} P_{k,j}(t) t^{k-1} d\sigma(t) P_{k,l}^*(t) = \begin{cases} 0_q & j \ne l \\ \widehat{H}_{k,j} & j = l \end{cases},\ k = 1, 2. \tag{14}$$

Let $P_{1,n}$, $P_{2,n}$ be OMP and be $Q_{1,n}$, $Q_{2,n}$ their second kind polynomials. The following identities hold:

$$U^{(2n)}(z) = \begin{pmatrix} Q^*_{2,n}(\bar{z})Q^{*^{-1}}_{2,n}(0) & -Q^*_{1,n}(\bar{z})P^{*^{-1}}_{1,n}(0) \\ -zP^*_{2,n}(\bar{z})Q^{*^{-1}}_{2,n}(0) & P^*_{1,n}(\bar{z})P^{*^{-1}}_{1,n}(0) \end{pmatrix}. \qquad (U2n)$$

If $\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$: $\frac{Q^*_{1,n}(\bar{z})}{P^*_{1,n}(\bar{z})} = -\cfrac{I_q}{-zM_0 + \cfrac{I_q}{L_0 + \cfrac{I_q}{\ddots + L_{n-2} + \cfrac{I_q}{-zM_{n-1} + L^{-1}_{n-1}}}}}$,

If $\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$: $\frac{Q^*_{2,n}(\bar{z})}{zP^*_{2,n}(\bar{z})} = -\cfrac{I_q}{-zM_0 + \cfrac{I_q}{L_0 + \cfrac{I_q}{\ddots - zM_{n-1} + \cfrac{I_q}{L_{n-1} - z^{-1}M^{-1}_n}}}}$.

ACR, On Dyukarev's resolvent matrix for a truncated Stieltjes matrix moment problem under the view of orthogonal matrix polynomials, Linear Algebra Appl. 474, 2015.

Nonnegative pair. Dyukarev 1981, 1999

Let $\mathbf{p}$ and $\mathbf{q}$ be $q \times q$ complex matrix-valued functions which are meromorphic in $\mathbb{C} \setminus [0, +\infty)$. Then $\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}$ is called a Stieltjes pair if there exists a discrete subset $\mathcal{D}_{pq}$ in $\mathbb{C} \setminus [0, \infty)$ such that the following conditions are satisfied:

(i) For $z \in \mathbb{C} \setminus ([0, +\infty) \cup \mathcal{D}_{pq})$,

$$\mathbf{p}^*(z)\mathbf{p}(z) + \mathbf{q}^*(z)\mathbf{q}(z) > 0. \tag{15}$$

(ii) For $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$

$$\frac{1}{2\mathrm{Im}\, z} \begin{pmatrix} \mathbf{p}(z) \\ \mathbf{q}(z) \end{pmatrix}^* (-J_q) \begin{pmatrix} \mathbf{p}(z) \\ \mathbf{q}(z) \end{pmatrix} \geq 0. \tag{16}$$

(iii) $\frac{1}{2\mathrm{Im}\, z} \begin{pmatrix} \mathbf{p}(z) \\ \mathbf{p}(z) \end{pmatrix}^* J_q^{\pi} \begin{pmatrix} \mathbf{p}(z) \\ \mathbf{q}(z) \end{pmatrix} \geq 0, \quad z \in \{z \in \mathbb{C} : \, Re\, z < 0\} \setminus \mathcal{D},$

where $J_q^{\pi} := \begin{pmatrix} 0_q & I_q \\ I_q & 0_q \end{pmatrix}$. The pairs $\begin{pmatrix} \mathbf{p}_1 \\ \mathbf{q}_1 \end{pmatrix}$ and $\begin{pmatrix} \mathbf{p}_2 \\ \mathbf{q}_2 \end{pmatrix}$ are said to be equivalent if there exists a matrix function $\mathbf{Q}(z)$ such that the matrices $\mathbf{Q}(z)$, $\mathbf{Q}^{-1}(z)$ are both meromorphic in $\mathbb{C} \setminus [0, +\infty)$ and $\mathbf{p}_1 = \mathbf{p}_2\mathbf{Q}, \quad \mathbf{q}_1 = \mathbf{q}_2\mathbf{Q}$. Let $\mathcal{S}_{\infty}$ denote the set of equivalence classes of Stieltjes pairs.

Let $P_{1,n}$, $P_{2,n}$ be OMP and be $Q_{1,n}$, $Q_{2,n}$ their second kind polynomials. The following identities hold:

$$U^{(2n)}(z) = \begin{pmatrix} Q_{2,n}^*(\bar{z}) {Q_{2,n}^*}^{-1}(0) & -Q_{1,n}^*(\bar{z}) {P_{1,n}^*}^{-1}(0) \\ -zP_{2,n}^*(\bar{z}) {Q_{2,n}^*}^{-1}(0) & P_{1,n}^*(\bar{z}) {P_{1,n}^*}^{-1}(0) \end{pmatrix}. \qquad (U2n)$$

If $\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$:
$$\frac{Q_{1,n}^*(\bar{z})}{P_{1,n}^*(\bar{z})} = -\cfrac{I_q}{-zM_0 + \cfrac{I_q}{L_0 + \cfrac{I_q}{\ddots + L_{n-2} + \cfrac{I_q}{-zM_{n-1} + L_{n-1}^{-1}}}}},$$

If $\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$:
$$\frac{Q_{2,n}^*(\bar{z})}{zP_{2,n}^*(\bar{z})} = -\cfrac{I_q}{-zM_0 + \cfrac{I_q}{L_0 + \cfrac{I_q}{\ddots - zM_{n-1} + \cfrac{I_q}{L_{n-1} - z^{-1}M_n^{-1}}}}}.$$

ACR, On Dyukarev's resolvent matrix for a truncated Stieltjes matrix moment problem under the view of orthogonal matrix polynomials, Linear Algebra Appl. 474, 2015.

Proposition 2

Let $(L_j)_{j=0}^{m-1}$ and $(M_j)_{j=0}^{m-1}$ (resp. $(L_j)_{j=0}^{m}$ and $(M_j)_{j=0}^{m-1}$) be two sequences of positive Hermitian complex $q \times q$ matrices. Let the sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m+1}$) be recursively defined by

$$s_{2j} := \begin{cases} M_0^{-1}, & \text{if } j = 0 \\ Y_{1,j}^* H_{1,j-1}^{-1} Y_{1,j} + (\overrightarrow{\prod}_{k=0}^{j-1} M_k L_k)^{-1^*} M_k^{-1} (\overrightarrow{\prod}_{k=0}^{j-1} M_k L_k)^{-1} & \text{if } j \geq 1 \end{cases} \tag{17}$$

and

$$s_{2j+1} := \begin{cases} (M_0 L_0)^{-1^*} L_0 (M_0 L_0)^{-1}, & \text{if } j = 0 \\ Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j} + (\overrightarrow{\prod}_{k=0}^{j} M_k L_k)^{-1^*} L_k \left(\overrightarrow{\prod}_{k=0}^{j} M_k L_k\right)^{-1} & \text{if } j \geq 1 \end{cases} \tag{18}$$

for $j = 0, \ldots, 2m$ (resp. $j = 0, \ldots, 2m+1$). Then $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m+1}$) is a Stieltjes positive sequence.

$$\widehat{H}_{1,0} = M_0^{-1}, \ \widehat{H}_{1,j} = \left(\overrightarrow{\prod}_{k=0}^{j-1} M_k L_k\right)^{-1^*} M_k^{-1} \left(\overrightarrow{\prod}_{k=0}^{j-1} M_k L_k\right)^{-1}, \tag{19}$$

$$\widehat{H}_{2,0} = M_0^{-1^*} L_0^{-1} M_0^{-1}, \ \widehat{H}_{2,j} = \left(\overrightarrow{\prod}_{k=0}^{j} M_k L_k\right)^{-1^*} L_k \left(\overrightarrow{\prod}_{k=0}^{j} M_k L_k\right)^{-1}. \tag{20}$$

ACR, On Dyukarev's Resolvent Matrix for a Truncated Stieltjes Matrix Moment Problem under the View of Orthogonal Matrix Polynomials, Linear Algebra Appl., 474, 2015.

Since $L_j > 0$ and $M_j > 0$, we get that

$$\widehat{H}_{1,j} > 0, \qquad \widehat{H}_{2,j} > 0, \quad \text{for all} \quad j \in \mathbb{N}_0. \tag{21}$$

Let

$$H_{1,j} = \begin{pmatrix} H_{1,j-1} & Y_{1,j} \\ Y_{1,j}^* & s_{2j} \end{pmatrix}, \quad H_{2,j-1} = \begin{pmatrix} H_{2,j-2} & Y_{2,j-1} \\ Y_{2,j-1}^* & s_{2j-1} \end{pmatrix}. \tag{22}$$

By using (21), (22), we obtain $H_{1,m}$ which is positive definite; (resp. $H_{2,m-1}$ is positive definite). Consequently, the sequence $(s_j)_{j=0}^{2m}$ is a Stieltjes positive sequence.

Lemma 1

Let $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ be a Hermitian $(n+m) \times (n+m)$ matrix. Then the following statements are equivalent:

a) $A > 0$.

b) $A_{11} > 0$ and $A_{22} - A_{12}^* A_{11}^{-1} A_{12} > 0$.

c) $A_{22} > 0$ and $A_{11} - A_{12} A_{22}^{-1} A_{12}^* > 0$.

Example 1. Let $c_j = 1$ for $0 \le j \le 2$, $d_j = \frac{1}{j+1}$ for $0 \le j \le 2$. Find a positive sequence $(s_j)_{j=0}^{5}$.

$$\cfrac{1}{-z + \cfrac{1}{1 + \cfrac{1}{-z + \cfrac{1}{\frac{1}{2} + \cfrac{1}{-z + (\frac{1}{3})^{-1}}}}}}$$

The moments are $s_j = j!$ for $0 \le j \le 5$,

$P_{1,0}(z) = 1,\ P_{1,1}(z) = z - 1,$

$P_{1,2}(z) = z^2 - 4z + 2,$

$P_{1,3}(z) = z^3 - 9z^2 + 18z - 6,$

$Q_{1,0}(z) = 0,\ Q_{1,1}(z) = 1,$

$Q_{1,2}(z) = z - 3,\ Q_{1,3}(z) = z^2 - 8z^2 + 11.$

We used the following relations:

$$\widehat{H}_{1,0} = c_0^{-1}, \quad \widehat{H}_{1,j} = \left(\prod_{k=0}^{j-1} c_k d_k\right)^{-1*} c_j^{-1} \left(\prod_{k=0}^{j-1} c_k d_k\right)^{-1}, \tag{23}$$

$$\widehat{H}_{2,0} = c_0^{-1} d_0^{-1} c_0^{-1}, \quad \widehat{H}_{2,j} = \left(\prod_{k=0}^{j} c_k d_k\right)^{-1*} d_j^{-1} \left(\prod_{k=0}^{j} c_k d_k\right)^{-1}. \tag{24}$$

ACR, On Dyukarev's resolvent matrix for a truncated Stieltjes matrix moment problem under the view of orthogonal matrix polynomials, Linear Algebra Appl. 474, 2015.

To prove the equalities

$$\widehat{H}_{1,0} = c_0^{-1}, \quad \widehat{H}_{1,j} = \left(\prod_{k=0}^{j-1} c_k d_k\right)^{-1^*} c_j^{-1} \left(\prod_{k=0}^{j-1} c_k d_k\right)^{-1}, \tag{25}$$

$$\widehat{H}_{2,0} = c_0^{-1} d_0^{-1} c_0^{-1}, \quad \widehat{H}_{2,j} = \left(\prod_{k=0}^{j} c_k d_k\right)^{-1^*} d_j^{-1} \left(\prod_{k=0}^{j} c_k d_k\right)^{-1}, \tag{26}$$

one uses Dyukarev's representation:

$$U^{(2j)}(z) = \begin{pmatrix} I_q & 0_q \\ -zM_0 & I_q \end{pmatrix} \begin{pmatrix} I_q & L_0 \\ 0_q & I_q \end{pmatrix} \cdots \begin{pmatrix} I_q & L_{j-1} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -zM_j & I_q \end{pmatrix}, \tag{27}$$

$$U^{(2j+1)}(z) = \begin{pmatrix} I_q & 0_q \\ -zM_0 & I_q \end{pmatrix} \begin{pmatrix} I_q & L_0 \\ 0_q & I_q \end{pmatrix} \cdots \begin{pmatrix} I_q & 0_q \\ -zM_j & I_q \end{pmatrix} \begin{pmatrix} I_q & L_j \\ 0_q & I_q \end{pmatrix} \tag{28}$$

Matrix moments $s_j(\alpha)$ by using the Laplace transform

Let σ be a monotone nondecreasing $q \times q$ matrix function $\sigma : [0, +\infty) \to \mathbb{R}$ satisfying $\sigma(x + 0) = \sigma(x)$ for $x \in [0, +\infty)$. For $\alpha \geq 0$, we define the matrix moments $s_j(\alpha)$ by using the Laplace transform

$$s_j(\alpha) := \int_0^\infty x^j e^{-\alpha x} d\sigma(x), \quad j \in \mathbb{N} \cup \{0\}, \tag{29}$$

Furthermore, in the sequel we assume that the Hankel block matrices

$$H_{1,j}(\alpha) := \begin{pmatrix} s_0(\alpha) & s_1(\alpha) & \dots & s_j(\alpha) \\ s_1(\alpha) & s_2(\alpha) & \dots & s_{j+1}(\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ s_j(\alpha) & s_{j+1}(\alpha) & \dots & s_{2j}(\alpha) \end{pmatrix}, H_{2,j}(\alpha) := \begin{pmatrix} s_1(\alpha) & s_2(\alpha) & \dots & s_{j+1}(\alpha) \\ s_2(\alpha) & s_3(\alpha) & \dots & s_{j+1}(\alpha) \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+1}(\alpha) & s_{j+1}(\alpha) & \dots & s_{2j+1}(\alpha) \end{pmatrix} \tag{30}$$

are both positive definite for $j \geq 0$ and $\alpha \in [0, +\infty)$.

Example of a matrix distribution on $[0, +\infty)$

Consider the 2×2 matrix distribution on $[0, +\infty)$,

$$\sigma(x) = \begin{pmatrix} 4 - 2e^{-\frac{1}{2}x} & e^{-x} \\ e^{-x} & 2 - e^{-x} \end{pmatrix}, \tag{31}$$

which corresponds to a positive matrix measure on $[0, +\infty)$. The matrices

$$s_j(\alpha) = j! \begin{pmatrix} \frac{1}{(\frac{1}{2}+\alpha)^{j+1}} & -\frac{1}{(1+\alpha)^{j+1}} \\ -\frac{1}{(1+\alpha)^{j+1}} & \frac{1}{(1+\alpha)^{j+1}} \end{pmatrix}, \quad j \geq 0 \tag{32}$$

are the corresponding moments (33). One can immediately verify that the block matrices $H_{1,j}$ and $H_{2,j}$ for $j = 0, 1, 2 \ldots$ are positive definite matrices.

- C. Berg, J. P. R. Christensen, P. Ressel, Positive definite functions on Abelian semigroups, Math. Ann. 223, 1976.

Stieltjes transform of $e^{-\alpha x}\sigma(dx)$

Definition 3

A sequence of matrix moments $(s_j(\alpha))_{j=0}^{\infty}$ is called a Stieltjes positive definite sequence if the Hankel block matrices $H_{1,j}(\alpha)$ and $H_{2,j}(\alpha)$ defined as in (30) are positive definite for $j \geq 0$ and $\alpha \in [0, +\infty)$.

Given a sequence $(s_j(\alpha))_{j\geq 0}$ of $q \times q$ matrices, find the set $\mathcal{M}_\alpha$ of positive measures $e^{-\alpha x}\sigma(dx)$ for x belonging to $[0, +\infty)$ and $\alpha \in [0, +\infty)$ such that (29) holds for $j \geq 0$.
Let $e^{-\alpha x}\sigma(dx) \in \mathcal{M}_\alpha$. The function

$$s(z, \alpha) := \int_{[0,+\infty)} \frac{e^{-x\alpha}\sigma(dx)}{x - z}, \quad z \in \mathbb{C} \setminus [0, +\infty) \tag{33}$$

is called the Stieltjes transform of $e^{-\alpha x}\sigma(dx)$.

The asymptotic relation between the Stieltjes transform $s(z,\alpha)$ and the moments $s_j(\alpha)$ near the point $z=+\infty$ reads

$$s(z,\alpha) = -\frac{s_0(\alpha)}{z} - \frac{s_1(\alpha)}{z^2} - \ldots - \frac{s_j(\alpha)}{z^{j+1}} - \ldots . \tag{34}$$

From (33) for $[0,+\infty)$, we get the following obvious equality:

$$\dot{s}_j(\alpha) = -s_{j+1}(\alpha). \tag{35}$$

On other hand, by employing (34) and (35) we attain the following relation:

$$\dot{s}(z,\alpha) = -s_0(\alpha) - zs(z,\alpha).$$

Further properties of the Stieltjes transform $s(z,\alpha)$ will be studied elsewhere.

For $\alpha \in [0, +\infty)$, let $\widehat{H}_{1,j}$ (resp. $\widehat{H}_{2,j}$) denote the Schur complement of the block $H_{1,j-1}$ in $H_{1,j}$ (resp. of the block $H_{2,j-1}$ in $H_{2,j}$):

$$\widehat{H}_{1,0}(\alpha) := s_0(\alpha), \quad \widehat{H}_{1,j}(\alpha) := s_{2j}(\alpha) - Y_{1,j}^*(\alpha) H_{1,j-1}^{-1}(\alpha) Y_{1,j}(\alpha), \quad j \geq 1, \tag{36}$$

$$\widehat{H}_{2,0}(\alpha) := s_1(\alpha), \quad \widehat{H}_{2,j}(\alpha) := s_{2j+1}(\alpha) - Y_{2,j}^*(\alpha) H_{2,j-1}^{-1}(\alpha) Y_{2,j}(\alpha), \quad j \geq 1. \tag{37}$$

These matrices are positive definite matrices, as well as the matrices $H_{1,j}$ (resp. $H_{2,j}$). In the scalar case, the matrices $\widehat{H}_{1,j}$ and $\widehat{H}_{1,j}$ have the form

$$\widehat{H}_{1,j} = \frac{|H_{1,j}|}{|H_{1,j-1}|}, \qquad \widehat{H}_{2,j} = \frac{|H_{2,j}|}{|H_{2,j-1}|}, \tag{38}$$

where $|\widehat{H}_{r,j}|$ denotes the determinant of $\widehat{H}_{r,j}$. Equality (38) is readily proved by calculating the determinant of the Schur complement $\widehat{H}_{r,j}$.

Definition 4

Let $\widehat{H}_{1,j}$ and $\widehat{H}_{2,j}$ be as in (36) and (37), respectively. For $\alpha \in [0, +\infty)$, define

$$A_{1,0}(\alpha) := \widehat{H}_{2,0}(\alpha)\widehat{H}_{1,0}^{-1}(\alpha), \tag{39}$$

$$A_{1,j}(\alpha) := \widehat{H}_{2,j}(\alpha)\widehat{H}_{1,j}^{-1}(\alpha) + \widehat{H}_{1,j}(\alpha)\widehat{H}_{2,j-1}^{-1}(\alpha), \ j \geq 1, \tag{40}$$

$$A_{2,j}(\alpha) := \widehat{H}_{1,j+1}(\alpha)\widehat{H}_{2,j}^{-1}(\alpha) + \widehat{H}_{2,j}(\alpha)\widehat{H}_{1,j}^{-1}(\alpha), \ j \geq 0. \tag{41}$$

For $r = 1, 2$, $j \geq 0$ and $\alpha \in [0, +\infty)$, denote

$$B_{r,j}(\alpha) := \widehat{H}_{r,j}^{-1}(\alpha)\widehat{H}_{r,j+1}(\alpha). \tag{42}$$

The matrix coefficients $A_j(\alpha)$ and $B_j(\alpha)$ appear in the discrete linear system

$$\begin{aligned} P_{j+1}(x,\alpha) =& (xI_q - A_j(\alpha))P_j(x,\alpha) - B_{j-1}(\alpha)^* P_{j-1}(x,\alpha), \quad j \geq 1, \\ P_0(x,\alpha) =& I_q, \quad P_1(x,\alpha) = xI_q - A_0(\alpha), \end{aligned}$$

for $\alpha \in [0, +\infty)$, where I_q is the $q \times q$ unit matrix and the asterisk denotes the matrix adjoint.

- ACR, The matrix Toda equations for coefficients of a matrix three-term recurrence relation, Operators and Matrices, 13(4), 2019 .

In the scalar case, the matrices $A_{r,j}$ and $B_{r,j}$ have the form $A_{1,0} = \frac{s_1}{s_0}$, $A_{2,0} = \frac{s_2}{s_1}$:

$$A_{1,1} = \frac{|H_{2,1}|H_{1,0}}{H_{2,0}|H_{1,1}|} + \frac{|H_{1,1}|}{H_{1,0}H_{2,0}},$$
$$A_{1,j} = \frac{|H_{2,j}||H_{1,j-1}|}{|H_{2,j-1}||H_{1,j}|} + \frac{|H_{1,j}||H_{2,j-2}|}{|H_{1,j-1}||H_{2,j-1}|},$$
$$A_{2,j} = \frac{|H_{1,j+1}||H_{2,j-1}|}{|H_{1,j}||H_{2,j}|} + \frac{|H_{2,j}||H_{1,j-1}|}{|H_{2,j-1}||H_{1,j}|}$$

for $j \geq 1$. Furthermore, $B_{1,0} = \frac{|H_{1,1}|}{s_0^2}$, $B_{2,0} = \frac{|H_{2,1}|}{s_1^2}$ and $B_{r,j} = \frac{|H_{r,j-1}||H_{r,j+1}|}{|H_{r,j}|^2}$ for $j \geq 1$.

The Toda matrix equation

Lemma 2

Let $\widehat{H}_{r,j}$ for $r = 1, 2$ be as in (36) and (37) and $\alpha \in (0, +\infty)$, then following equalities are valid:

$$\dot{\widehat{H}}_{1,j} = -\widehat{H}_{2,j} - \widehat{H}_{1,j}\widehat{H}_{2,j-1}^{-1}\widehat{H}_{1,j}, \quad (43)$$

$$\dot{\widehat{H}}_{2,j} = -\widehat{H}_{1,j+1} - \widehat{H}_{2,j}\widehat{H}_{1,j}^{-1}\widehat{H}_{2,j}. \quad (44)$$

Theorem 3

Let $A_{r,j}$ and $B_{r,j}$ be as in Definition 1. For $r = 1, 2$, the following identities are valid:

$$\dot{A}_{r,0} = -B_{r,0}^*, \quad (45)$$

$$\dot{A}_{r,j} = B_{r,j-1}^* - B_{r,j}^*, \quad (46)$$

$$\dot{B}_{r,j}^* = B_{r,j}^* A_{r,j} - A_{r,j+1} B_{r,j}^* \quad (47)$$

for $j \geq 1$ and $\alpha \in (0, +\infty)$.

- ACR, The matrix Toda equations for coefficients of a matrix three-term recurrence relation, Operators and Matrices, 13(4), 2019.

Matrix polynomial $\mathbf{f}_n(\alpha, z) := A_0(\alpha)z^n + A_1(\alpha)z^{n-1} + \ldots + A_n(\alpha)$ (*)

$\deg \mathbf{f}_n = n$ if A_0 is nonzero and $\alpha \in [0, \infty)$. We suppose that $\det A_0 \neq 0$. Let

$$\mathbf{h}_n(\alpha, u) := \begin{cases} A_0 u^m + A_2 u^{m-1} + \ldots + A_{2m}, & n = 2m, \\ A_1 u^m + A_3 u^{m-1} + \ldots + A_{2m+1}, & n = 2m+1, \end{cases} \tag{48}$$

$$\mathbf{g}_n(\alpha, u) := \begin{cases} A_1 u^{m-1} + A_3 u^{m-2} + \ldots + A_{2m-1}, & n = 2m, \\ A_0 u^m + A_2 u^{m-1} + \ldots + A_{2m}, & n = 2m+1. \end{cases} \tag{49}$$

Clearly $\mathbf{f}_n(z) = \mathbf{h}_n(z^2) + z\,\mathbf{g}_n(z^2)$. For A, $B \in \mathbb{C}^{q\times q}$ with B invertible, $\frac{A}{B} := AB^{-1}$.

Definition 5. The $q \times q$ matrix polynomial $\mathbf{f}_n$ in (*) is called a matrix Hurwitz type polynomial if there exist two sequences of $q \times q$ positive definite matrices $(\mathbf{c}_k)_{k=0}^{m-1}$ and $(\mathbf{d}_k)_{k=0}^{m-1}$ ($(\mathbf{c}_k)_{k=0}^{m}$ and $(\mathbf{d}_k)_{k=0}^{m-1}$) such that,

$$\frac{\mathbf{g}_n(u)}{\mathbf{h}_n(u)} = \cfrac{I_q}{u\mathbf{c}_0 + \cfrac{I_q}{\mathbf{d}_0 + \cfrac{I_q}{\ddots + \mathbf{d}_{m-2} + \cfrac{I_q}{+u\mathbf{c}_{m-1} + \mathbf{d}_{m-1}^{-1}}}}}, \tag{50}$$

for $n = 2m$, for all $u \in \mathbb{C}$ with $\det \mathbf{h}(u) \neq 0$ and,

$$\frac{\mathbf{h}_n(u)}{u\mathbf{g}_n(u)} = \cfrac{I_q}{u\mathbf{c}_0 + \cfrac{I_q}{\mathbf{d}_0 + \cfrac{I_q}{\ddots + u\mathbf{c}_{m-1} + \cfrac{I_q}{\mathbf{d}_{m-1} + u^{-1}\mathbf{c}_m^{-1}}}}} \tag{51}$$

for $n = 2m+1$, for all $u \in \mathbb{C} \setminus \{0\}$ with $\det \mathbf{g}(u) \neq 0$.

Lemma 3

Let $\mathbf{h}_n$ and $\mathbf{g}_n$ be defined as in (48) and (49). Let $|u_0| \geq \max\{u : \det \mathbf{h}_n(u) = 0\}$ (resp. $|u_0| \geq \max\{u : \det \mathbf{g}_n(u) = 0\}$). For $|u| > |u_0|$, the power series expansion of $\mathbf{g}_n/\mathbf{h}_n$ (resp. $\mathbf{h}_n/\mathbf{g}_n$) in negative powers of z,

$$\frac{\mathbf{g}_n(u)}{\mathbf{h}_n(u)} = \frac{s_0}{u} - \frac{s_1}{z^2} + \ldots + (-1)^n \frac{s_n}{u^{n+1}} + \ldots, \quad n = 2m, \qquad (20)$$

$$\frac{\mathbf{h}_n(u)}{\mathbf{g}_n(u)} = s_0 - \frac{s_1}{u} + \ldots + (-1)^n \frac{s_n}{u^n} + \ldots, \quad n = 2m+1. \qquad (21)$$

Then

$$A^{[m-1,0]} := \begin{pmatrix} A_{2m-1} \\ A_{2m-3} \\ \ldots \\ A_1 \end{pmatrix} = \begin{pmatrix} s_0 & -s_1 & \ldots & (-1)^{m-1}s_{m-1} \\ \ldots & \ldots & \ldots & \ldots \\ 0_q & \ldots & \ldots & -s_1 \\ 0_q & 0_q & \ldots & s_0 \end{pmatrix} \begin{pmatrix} A_{2m-2} \\ A_{2m-4} \\ \ldots \\ A_0 \end{pmatrix} = S_{[0,m-1]} J_m A_{[m-1,0]},$$

$$Y_{1,m}A_0 = \begin{pmatrix} s_m A_0 \\ s_{m+1}A_0 \\ \ldots \\ s_{2m-1}A_0 \end{pmatrix} = \begin{pmatrix} s_0 & s_1 & \ldots & s_{m-1} \\ s_1 & s_0 & \ldots & s_{m-2} \\ \ldots & \ldots & \ldots & \ldots \\ s_{m-1} & s_m & \ldots & s_{2m-2} \end{pmatrix} \begin{pmatrix} (-1)^{m-1}A_{2m} \\ (-1)^{m-2}A_{2m-2} \\ \ldots \\ A_2 \end{pmatrix} = H_{1,m-1} J_m A_{m,1}.$$

Consequently for $A_0 = I_q$.
$\mathbf{h}_n(u) = (-1)^m P^*_{1,m}(-\bar{u})$ and $\mathbf{g}_n(u) = (-1)^{m+1} Q^*_{1,m}(-\bar{u})$, $n = 2m$,
$\mathbf{g}_n(u) = (-1)^m P^*_{2,m}(-\bar{u})$, and $\mathbf{h}_n(u) = (-1)^m Q^*_{2,m}(-\bar{u})$, $n = 2m+1$.

Lemma 4

Let $\mathbf{f}_n$ be a matrix Hurwitz type polynomial then there exists a positive sequence $(s_j)_{j=0}^{2n}$ $((s_j)_{j=0}^{2n+1})$ such that

$$\mathbf{f}_n(z) = \begin{cases} (-1)^m P_{1,m}^*(-\bar{z}^2) + (-1)^{m+1} z\, Q_{1,m}^*(-\bar{z}^2), & n = 2m, \\ (-1)^m Q_{2,m}^*(-\bar{z}^2) + (-1)^m z\, P_{2,m}^*(-\bar{z}^2), & n = 2m+1 \end{cases} \tag{52}$$

where $P_{1,m}$, $Q_{1,m}$ are constructed by $((s_j)_{j=0}^{2n+1})$ ($P_{2,m}$, $Q_{2,m}$ are constructed by $((s_j)_{j=0}^{2n})$).

$$\frac{\mathbf{g}_n(u)}{\mathbf{h}_n(u)} = \cfrac{I_q}{u\mathbf{c}_0 + \cfrac{I_q}{\mathbf{d}_0 + \cfrac{I_q}{\ddots + \mathbf{d}_{m-2} + \cfrac{I_q}{+u\mathbf{c}_{m-1} + \mathbf{d}_{m-1}^{-1}}}}},$$

for $n = 2m$, for all $u \in \mathbb{C}$ with $\det \mathbf{h}(u) \neq 0$ and,

By Proposition 2 we recover the positive sequence $(s_j)_{j=0}^{2n}$ from the sequence $((\mathbf{c}_j)_{j=0}^{m}, (\mathbf{d}_j)_{j=0}^{m})$.

By using the positive sequence $(s_j)_{j=0}^{2n}$ we construct $P_{1,m}$ and $Q_{1,m}$.

By Lemma 3 we get that $f_n(z)$ is equal to

$$(-1)^m P_{1,m}^*(-\bar{z}^2) + (-1)^{m+1} z\, Q_{1,m}^*(-\bar{z}^2).$$

The Lemma is proved.

Lemma 5

Let $P_{1,m}$, $Q_{1,m}$ ($P_{2,m}$, $Q_{2,m}$) be a member of OMP of $[0,\infty)$ and its second kind polynomial, respectively, constructed by the positive sequences $(s_j)_{j=0}^{2n-1}$ ($(s_j)_{j=0}^{2n}$). Then the polynomial

$$\mathbf{f}_n(z) = \begin{cases} (-1)^m P^*_{1,m}(-\bar{z}^2) + (-1)^{m+1} z\, Q^*_{1,m}(-\bar{z}^2), & n = 2m, \\ (-1)^m Q^*_{2,m}(-\bar{z}^2) + (-1)^m t\, P^*_{2,m}(-\bar{z}^2), & n = 2m+1 \end{cases} \quad (53)$$

is a Hurwitz type polynomial.

For $(s_j)_{j=0}^{2n-1}$:

$$\frac{Q^*_{1,n}(\bar{z})}{P^*_{1,n}(\bar{z})} = -\cfrac{I_q}{-zM_0 + \cfrac{I_q}{L_0 + \cfrac{I_q}{\ddots + L_{n-2} + \cfrac{I_q}{-zM_{n-1} + L_{n-1}^{-1}}}}},$$

For $(s_j)_{j=0}^{2n}$:

$$\frac{Q^*_{2,n}(\bar{z})}{zP^*_{2,n}(\bar{z})} = -\cfrac{I_q}{-zM_0 + \cfrac{I_q}{L_0 + \cfrac{I_q}{\ddots - zM_{n-1} + \cfrac{I_q}{L_{n-1} - z^{-1}M_n^{-1}}}}}.$$

From Lemma 4 and Lemma 5, we obtain the following result.

Theorem 3

Let $P_{1,m}$, $Q_{1,m}$ ($P_{2,m}$, $Q_{2,m}$) be a member of OMP of $[0,\infty)$ and its second kind polynomial, respectively, constructed by the positive sequences $(s_j(\alpha))_{j=0}^{2n}$ $((s_j(\alpha))_{j=0}^{2n+1})$. Thus, for $\alpha \in [0,+\infty)$ f_n is a Hurwitz type matrix polynomial if and only if f_n admits the following presentation:

$$f_n(\alpha,z) = \begin{cases} (-1)^m(P_{1,m}(\alpha,-\bar{z}^2) - z\,Q_{1,m}(\alpha,-\bar{z}^2)), & n=2m, \\ (-1)^m(Q_{2,m}(\alpha,-\bar{z}^2) + z\,P_{2,m}(\alpha,-\bar{z}^2)), & n=2m+1. \end{cases} \tag{54}$$

Consider the differential equation $\dot{x} = Ax$, $x \in \mathbb{R}^n$ (**). This system is asymptotically stable in the Lyapunov sense if and only if all zeros of the characteristic polynomial $f_A(\lambda) := \det(\lambda I - A)$ have a negative real part.

Theorem 4

The system (**) is asymptotically stable if and only if the characteristic polynomial f_A

$$f_A(z) = \begin{cases} p_{1,m}(-z^2) - z\,q_{1,m}(-z^2), & n=2m, \\ q_{2,m}(-z^2) + z\,p_{2,m}(-z^2), & n=2m+1, \end{cases} \tag{55}$$

where $p_{1,m}$ (resp. $p_{2,m}$) is a scalar monic orthogonal polynomial on $[0,\infty)$ with respect to a positive measure σ (resp. $t\sigma$) on $[0,\infty)$, and $q_{1,m}$ and $q_{2,m}$ are their corresponding second kind polynomials.

Hurwitz matrix polynomials

- A matrix polynomial $F(z)$ is Hurwitz stable if the set of all zeros of $\det F(z)$ belongs to left half of the complex plane.
- A matrix Hurwitz type polynomial is a Hurwitz polynomial. [X. Zhan, A. Dyachenko, J. Comput. Appl. Math. 383, 2021].
- The set of matrix Hurwitz type polynomials do not constitute the entire class of Hurwitz stable matrix polynomials. [ACR: Comments on the paper "On the relation between Hurwitz stability of matrix polynomials and matrix-valued Stieltjes functions," J. Comput. Appl. Math. 417 (2023) by X. Zhan and Y. Hu, submitted to J. Comput. Appl. Math. in 2023]

Example 2: Consider the matrix polynomial

$$\mathbf{f}_2(z) = Iz^2 + \begin{pmatrix} \frac{48}{5} & \frac{28i}{5} \\ 0 & 4 \end{pmatrix} z + \begin{pmatrix} \frac{74}{5} & \frac{64i}{5} \\ 0 & 2 \end{pmatrix} \quad (56)$$

$$\det \mathbf{f}_2(z) = z^4 + \frac{68z^3}{5} + \frac{276z^2}{5} + \frac{392z}{5} + \frac{148}{5}. \quad (57)$$

The approximate value of the roots of the polynomial $\det \mathbf{f}_2(z)$ are

$$z_1 = -7.6705,\ z_2 = -3.4142,\ z_3 = -1.9294,\ z_4 = -0.5857. \quad (58)$$

Thus, $\mathbf{f}_2(z)$ is a Hurwitz polynomial, but $\mathbf{f}_2(z)$ is not a matrix Hurwitz type polynomial.

Example 3. Assume that $\sigma'(x) = \begin{pmatrix} e^{-x/2} & -ie^{-x} \\ ie^{-x} & e^{-x} \end{pmatrix}$. The Markov parameters or moments are defined by $s_j(\alpha) = j! \begin{pmatrix} \left(\alpha + \frac{1}{2}\right)^{-(j+1)} & -i(\alpha+1)^{-(j+1)} \\ i(\alpha+1)^{-(j+1)} & (\alpha+1)^{-(j+1)} \end{pmatrix}$. The corresponding polynomials $P_{1,1}$ and $Q_{1,1}$ have the form

$$P_{1,1}(\alpha, z) = \begin{pmatrix} z - \frac{4\alpha+3}{2\alpha^2+3\alpha+1} & -\frac{2i}{2\alpha+1} \\ 0 & z - \frac{1}{\alpha+1} \end{pmatrix}, \quad Q_{1,1}(\alpha, z) = \begin{pmatrix} \frac{2}{2\alpha+1} & -\frac{i}{\alpha+1} \\ \frac{i}{\alpha+1} & \frac{1}{\alpha+1} \end{pmatrix}, \tag{59}$$

and

$$\begin{aligned} \mathbf{f}_2(\alpha, z) &= -\left(P^*_{1,1}(\alpha, -\bar{z}^2) - Q^*_{1,1}(\alpha, -\bar{z}^2)\right) \\ &= Iz^2 + \begin{pmatrix} \frac{2}{2\alpha+1} & -\frac{i}{\alpha+1} \\ \frac{i}{\alpha+1} & \frac{1}{\alpha+1} \end{pmatrix} z + \begin{pmatrix} \frac{4\alpha+3}{2\alpha^2+3\alpha+1} & 0 \\ \frac{2i}{2\alpha+1} & \frac{1}{\alpha+1} \end{pmatrix}. \end{aligned} \tag{60}$$

Observe that the determinant $\det \mathbf{f}_2(\alpha, z)$ has the form

$$\det \mathbf{f}_2(\alpha, z) = z^4 + \frac{\left(4\alpha^2 + 7\alpha + 3\right) z^3}{(\alpha+1)^2(2\alpha+1)} + \frac{\left(6\alpha^2 + 10\alpha + 5\right) z^2}{(\alpha+1)^2(2\alpha+1)} + \frac{(4\alpha+3)z}{(\alpha+1)^2(2\alpha+1)} + \frac{4\alpha+3}{(\alpha+1)^2(2\alpha+1)}. \tag{61}$$

The polynomial $\det \mathbf{f}_2(\alpha, z)$ is a Hurwitz polynomial for each $\alpha \in [0, +\infty)$.

- L. Lerer, M. Tismenetsky, The Bezoutian and the eigenvalue-separation problem for matrix polynomials, Integral Equations Oper. Theory 5, 1982.

References

- Yu. Dyukarev, *Indeterminacy criteria for the Stieltjes matrix moment problem* , Mathematical Notes, Vol. 75, 1–2, 2004.
- ACR, C. Maedler, On Hankel positive definite perturbations of Hankel positive definite sequences and interrelations to orthogonal matrix polynomials, Complex Anal. Oper. Theory 8(8), 2014.
- ACR, On matrix Hurwitz type polynomials and their interrelations to Stieltjes positive definite sequences and orthogonal matrix polynomials, Linear Algebra Appl. 476, 2015.
- ACR, On Dyukarev's Resolvent Matrix for a Truncated Stieltjes Matrix Moment Problem under the View of Orthogonal Matrix Polynomials, Linear Algebra Appl. 474, 2015.
- ACR, The matrix Toda equations for coefficients of a matrix three-term recurrence relation, Operators and Matrices, 13(4), 2019.

Thank you