# On damping a control system of arbitrary order with global aftereffect on a temporal tree

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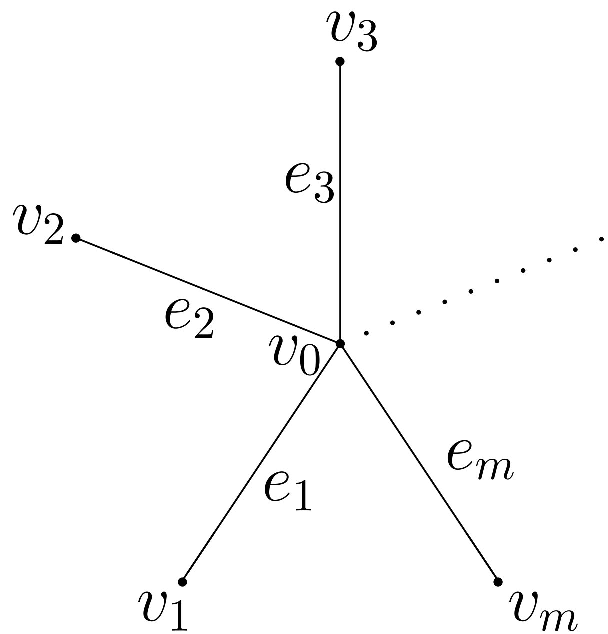


$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad 0 < x < 1, \quad j = \overline{1, m}, \tag{1}$$

where $q_j(x)$ are integrable functions,

$$y_1(1) = y_2(1) = \ldots = y_m(1) \quad \text{(continuity conditions)}, \tag{2}$$

$$y_1'(1) + y_2'(1) + \ldots + y_m'(1) = 0 \quad \text{(Kirchhoff's condition)}, \tag{3}$$

$$y_j(0) = 0, \quad j = \overline{1, m}. \tag{4}$$

Our goal was to define an operator with constant delay of the form

$$-y''(x) + q(x)y(x-a), \quad 0 < x < 1, \tag{5}$$

on graphs.

Various operators with deviating argument on an interval have been actively studied starting from the middle of the last century in connection with numerous applications. For example, see the monographs:


**[9]** Myshkis A.D. *Linear Differential Equations with a Delay Argument*, Nauka, Moscow, 1951.

**[10]** Bellman R. and Cooke K.L. *Differential-Difference Equations*, The RAND Corp. R-374-PR, 1963.

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**[12]** Hale J. *Theory of Functional-Differential Equations*, Springer-Verlag, NewYork, 1977.

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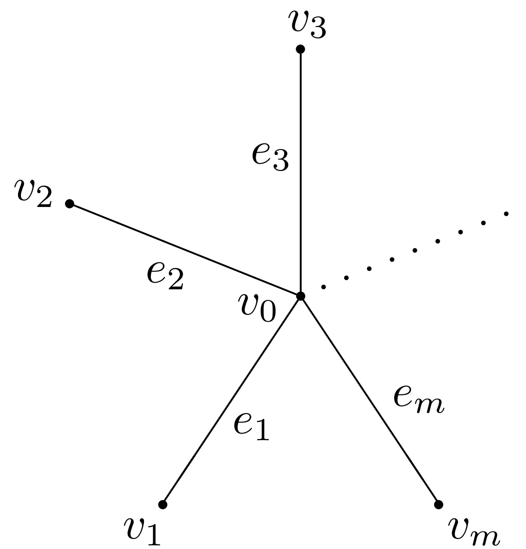
## Locally nonlocal case [14]

[14] Wang F. and Yang C.-F. *Traces for Sturm–Liouville operators with constant delays on a star graph*, Results Math. (2021) 76:220.




$$-y_j''(x) + q_j(x)y_j(x - a_j) = \lambda y_j(x), \quad 0 < x < 1, \quad j = \overline{1, m}, \tag{6}$$

where $q_j(x) = 0$ a.e. on $(0, a_j)$ for each $j$,

$$y_1(1) = y_2(1) = \ldots = y_m(1), \quad y_1'(1) + y_2'(1) + \ldots + y_m'(1) = 0, \tag{7}$$

$$y_j(0) = 0, \quad j = \overline{1, m}. \tag{4}$$

## Globally nonlocal case

$$-y''(x)+q(x)y(x-a)=\lambda y(x),\quad 0<x<2,\quad y(0)=y(2)=0,\tag{8}$$

$$a\in(0,2),\qquad q(x)=0\ \text{ a.e. on }\ (0,a).\tag{9}$$

For $x\in(0,1)$, we denote:

$y_1(x):=y(x),\ \ y_2(x):=y(x+1),\quad q_1(x):=q(x),\ \ q_2(x):=q(x+1).$

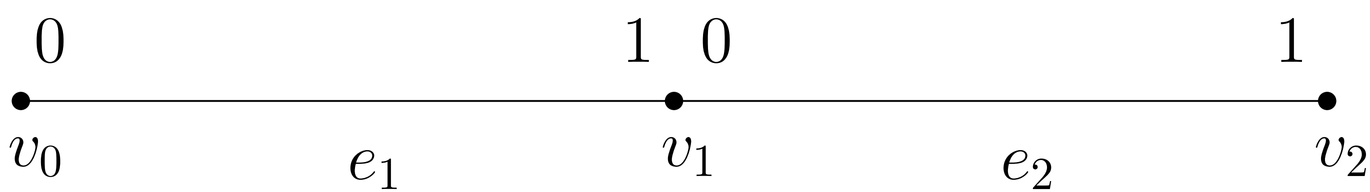
$$-y_j''(x)+q_j(x)y_j(x-a)=\lambda y_j(x),\quad 0<x<1,\quad j=1,2.\tag{10}$$

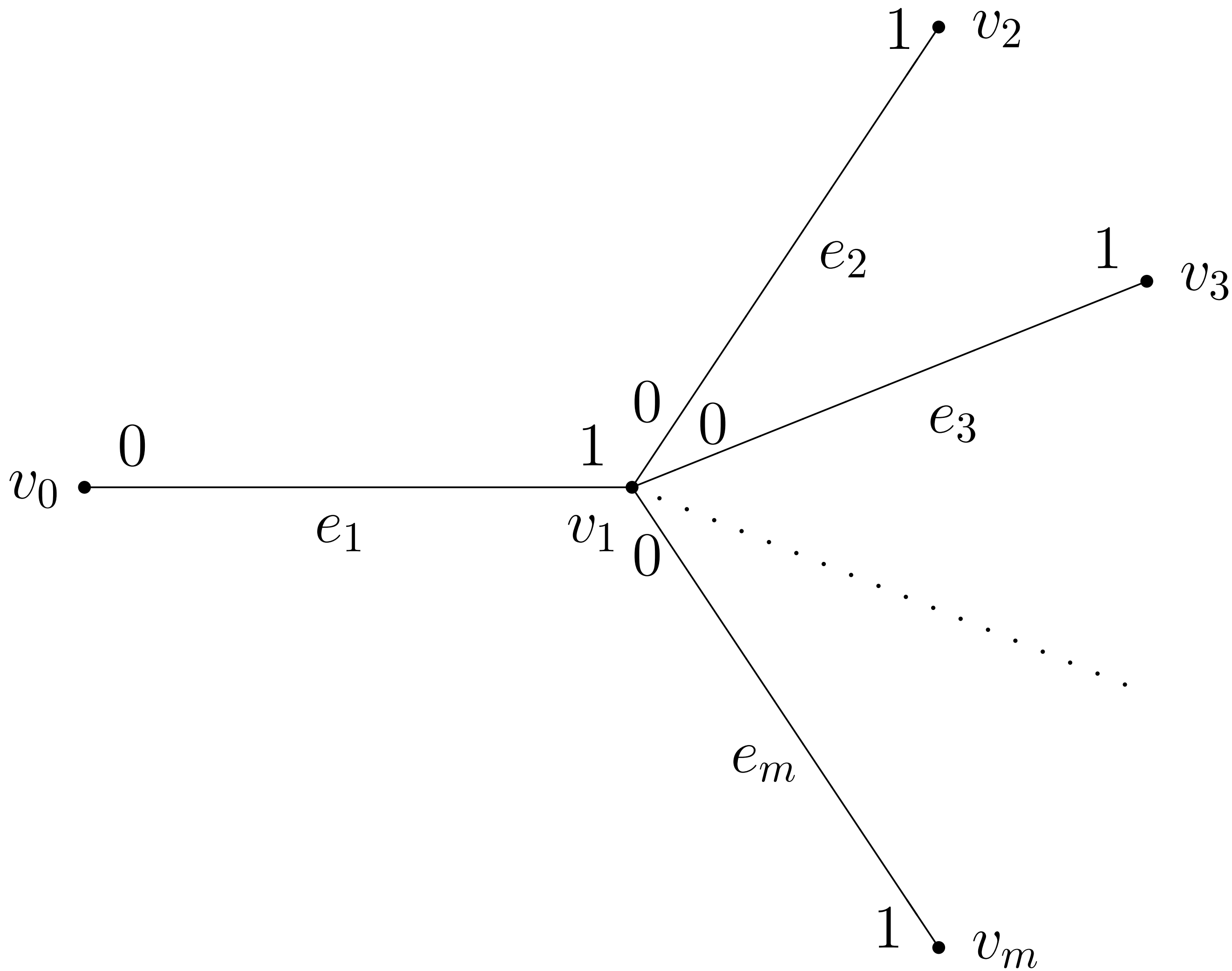
$$y_1(1)=y_2(0),\quad y_1'(1)=y_2'(0),\tag{11}$$

$$y_1(0)=y_2(1)=0,\tag{12}$$

$$y_2(x-a):=y_1(x-a+1),\quad \max\{0,a-1\}<x<\min\{a,1\},\tag{13}$$

$$q_1(x)=0,\ x\in(0,\min\{a,1\});\ \ q_2(x)=0,\ x\in(0,\max\{0,a-1\}).\tag{14}$$






**[15]** Buterin S.A. *Functional-differential operators on geometrical graphs with global delay and inverse spectral problems*, Res. Math. (2023) 78:79.

Let $a \in (0, 2)$. The corresponding BVP has the form:

$$-y_j''(x) + q_j(x)y_j(x-a) = \lambda y_j(x), \quad 0 < x < 1, \quad j = \overline{1,m}, \tag{15}$$

$$y_j(x-a) = y_1(x-a+1), \quad j = \overline{2,m}, \tag{16}$$

$$\text{for} \quad x \in \Big(\max\{0, a-1\}, \min\{a, 1\}\Big),$$

$$y_1(1) = y_2(0) = \ldots = y_m(0), \quad y_1'(1) = y_2'(0) + \ldots + y_m'(0), \tag{17}$$

$$y_1(0) = 0, \quad y_j(1) = 0, \quad j = \overline{2,m}. \tag{18}$$

Also assume that

$$\left.\begin{aligned} q_1(x) = 0 \ \text{ a.e. on } \ (0, \min\{a, 1\}), \\ q_j(x) = 0 \ \text{ a.e. on } \ (0, \max\{0, a-1\}), \ \ j = \overline{2,m}. \end{aligned}\right\} \tag{19}$$

[16] Buterin S. *On recovering St.–L.-type operators with global delay on graphs from two spectra*, Mathematics 11 (2023) no.12, art. no. 2688.

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Skubachevskii [18] (see also [13]) considered the following control system:

$$y'(t) + ay'(t-\tau) + by(t) + cy(t-\tau) = u(t), \quad t > 0. \tag{20}$$

where $a$, $b$, $c \in \mathbb{R}$ and $\tau > 0$ are fixed, $u(t)$ is a control function,

$$y(t) = \varphi(t) \in W_2^1[-\tau, 0], \quad t \in [-\tau, 0], \tag{21}$$

while $\varphi(t)$ is a known real-valued function.

**Problem**: *Bring (20) and (21) into the state $y(t) = 0$ for $t \geq T > 2\tau$.*

It is sufficient to find a control function $u(t) \in L_2(0, T)$ such that

$$y(t) = 0, \quad t \in [T-\tau, T]. \tag{22}$$

Thus, there arises the **variational problem** for energy functional

$$J(y) := \int_0^T \Big(y'(t) + ay'(t-\tau) + by(t) + cy(t-\tau)\Big)^2 \, dt \to \min \tag{23}$$

under the conditions (21) and (22).

**Theorem 1.** [18] *A function $y(t) \in W_2^1[-\tau, T]$ is a solution of the variational problem (21)–(23) if and only if $y(t)$ solves the boundary value problem for the equation*

$$\Big((1+a^2)y'(t) + ay'(t-\tau) + ay'(t+\tau)\Big)' + (c-ab)(y'(t-\tau) - y'(t+\tau))$$
$$= (b^2+c^2)y(t) + bc(y(t-\tau) + y(t+\tau)), \quad 0 < t < T - \tau, \qquad (24)$$

*under the conditions (21) and (22).*

The solution is understood in the generalized sense:

$$(1+a^2)y'(t) + ay'(t-\tau) + ay'(t+\tau) \in W_2^1[0, T-\tau]. \qquad (25)$$

**Theorem 2.** [18] *For any function $\varphi(t) \in W_2^1[-\tau, 0]$, there exists a unique generalized solution $y(t) \in W_2^1[-\tau, T]$, and*

$$\|y\|_{W_2^1[-\tau,T]} \leq c\|\varphi\|_{W_2^1[-\tau,0]},$$

*where $c$ does not depend on $\varphi(t)$.*

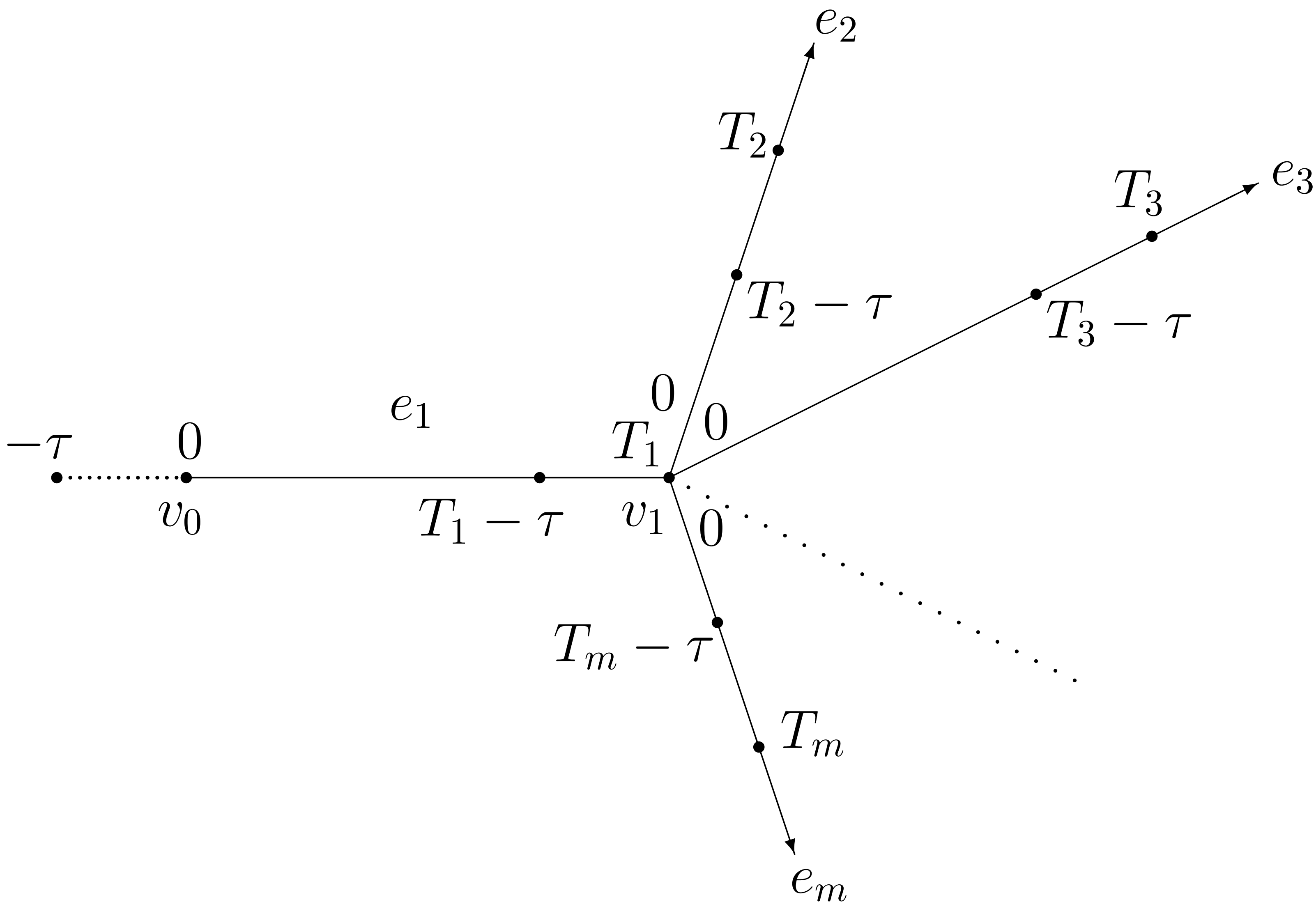



Fig. 1. A star-shaped graph $\Gamma_m$


**[21]** Buterin S. *On damping a control system with global aftereffect on quantum graphs*, arXiv:2308.00496 [math.OC], 2023.

Up to $t = T_1$, our control system is described by the equation ($\tau < T_1$)

$$\ell_1 y(t) := y_1'(t) + b_1 y_1(t) + c_1 y_1(t-\tau) = u_1(t), \quad 0 < t < T_1, \tag{26}$$

where $y_1(t)$ is defined on the edge $e_1 = [v_0, v_1]$ and has the prehistory

$$y_1(t) = \varphi(t) \in W_2^1[-\tau, 0], \quad t \in [-\tau, 0]. \tag{27}$$

At $v_1$, this system splits into $m-1$ independent processes described by

$$\ell_j y(t) := y_j'(t) + b_j y_j(t) + c_j y_j(t-\tau) = u_j(t), \quad t > 0, \quad j = \overline{2, m}, \tag{28}$$

but having a common history determined by (26), (27) and the relations

$$y_j(t) = y_1(t + T_1), \quad t \in (-\tau, 0), \quad j = \overline{2, m}. \tag{29}$$

Conditions (29) mean that the delay propagates through the internal vertex $v_1$. Besides (29), we impose the continuity conditions at $v_1$ :

$$y_j(0) = y_1(T_1), \quad j = \overline{2, m}. \tag{30}$$

Fix $T_j > \tau$ for $j = \overline{2, m}$. Then damping the control system (26)–(30) would mean that $y_j(t)$ becomes zero as soon as $t \geq T_j$ for $j = \overline{2, m}$. For ensuring this, it is sufficient to find $u_j(t)$, $j = \overline{1, m}$, that lead to

$$y_j(t) = 0, \quad t \in [T_j - \tau, T_j], \quad j = \overline{2, m}. \tag{31}$$

It can be interpreted as if all the "arisen" processes are just possible scenarios of one and the same process after the time point $t = T_1$.

In other words, at $t = T_1$, there appear $m - 1$ different scenarios of the further process flow determined, in turn, by different pairs of the coefficients $b_j$ and $c_j$ in equations (28).

Before $t = T_1$, there is no information about which scenario will be really fulfilled. Thus, the control functions $u_j(t)$, $j = \overline{1, m}$, should be chosen in such a way that the system will be damped surely at each possible outcome, i.e. on all the edges emanating from the vertex $v_1$.

Since the required controls $u_j(t)$ are far not unique, it is natural to try reducing the corresponding efforts $\|u_j\|_{L_2(0,T_j)}$ as much as possible.

When constructing the corresponding energy functional it is reasonable, as we demonstrate below, to regulate the entrance of each $\|u_j\|^2_{L_2(0,T_j)}$ by choosing a certain positive weight $\alpha_j$.

Thus, we arrive at the **variational problem**

$$\sum_{j=1}^{m} \alpha_j \int_0^{T_j} (\ell_j y(t))^2 \, dt \to \min \tag{32}$$

under the conditions (27) and (29)–(31), where $\alpha_j > 0$, $j = \overline{1,m}$.

**Th. 3.** *Functions* $y_1(t) \in W_2^1[-\tau, T_1]$, $y_j(t) \in W_2^1[0, T_j]$, $j = \overline{2,m}$, *form a solution of the variational problem (27), (29)–(32) if and only if they possess additional smoothness:* $y_1(t) \in W_2^2[0, T_1]$, $y_j(t) \in W_2^2[0, T_j - \tau]$, $j = \overline{2,m}$, *and solve the boundary value problem* $\mathcal{B}$ *(see the next slide).*

The problem $\mathcal{B}$ consists of the equations

$$\alpha_1(\ell_1 y)'(t) = \alpha_1 b_1 \ell_1 y(t) + \begin{cases} \alpha_1 c_1 \ell_1 y(t+\tau), & 0 < t < T_1 - \tau, \\ \sum_{\nu=2}^{m} \alpha_\nu c_\nu \ell_\nu y(t+\tau-T_1), & T_1 - \tau < t < T_1, \end{cases}$$

$$(\ell_j y)'(t) = b_j \ell_j y(t) + c_j \ell_j y_j(t+\tau), \quad 0 < t < T_j - \tau, \quad j = \overline{2, m},$$

along with all standing conditions (27), (29)–(31) as well as the new one

$$\alpha_1 y_1'(T_1) + \beta y_1(T_1) + \gamma y_1(T_1 - \tau) = \sum_{j=2}^{m} \alpha_j y_j'(0), \tag{33}$$

where

$$\beta := \alpha_1 b_1 - \sum_{j=2}^{m} \alpha_j b_j, \quad \gamma := \alpha_1 c_1 - \sum_{j=2}^{m} \alpha_j c_j. \tag{34}$$

**Theorem 4.** *The boundary value problem* $\mathcal{B}$ *has a unique solution. Moreover, the solution satisfies the estimate*

$$\|y_1\|_{W_2^1[0,T_1]} + \sum_{j=2}^{m} \|y_j\|_{W_2^1[0,T_j-\tau]} \le C\|\varphi\|_{W_2^1[-\tau,0]}, \qquad (35)$$

*where* $C$ *is independent of* $\varphi(t)$.

In [22], Theorems 3 and 4 were generalized to a control system of arbitrary order and neutral type with non-smooth complex coefficients on arbitrary tree.

**[22]** Buterin S.A. *On damping a control system of arbitrary order with global aftereffect on a tree*, Math. Notes 115 (2024) no. 6, 877–896.

**Example I.** Let $b_j,\ c_j$ and $T_j$ be independent of $j \in \{2, \dots, m\}$. Then we have $m-1$ copies of one and the same scenario starting from the time point $t = T_1$. Assume that

$$\alpha_1 = 1, \quad \alpha_j = \frac{1}{m-1}, \quad j = \overline{2, m}, \tag{36}$$

By the symmetry, the solution $[y_1, y_2, \dots, y_m]$ of the boundary value problem $\mathcal{B}$ contains $m-1$ equal components: $y_2(t) \equiv \dots \equiv y_m(t)$ (otherwise, this solution would not be unique). Hence, the number of equations in the problem $\mathcal{B}$ can be reduced to just two, namely:

$$(\ell_1 y)'(t) = b_1 \ell_1 y(t) + \begin{cases} c_1 \ell_1 y(t+\tau), & 0 < t < T_1 - \tau, \\ c_2 \ell_2 y(t + \tau - T_1), & T_1 - \tau < t < T_1, \end{cases} \tag{37}$$

and

$$(\ell_2 y)'(t) = b_2 \ell_2 y(t) + c_2 \ell_2 y_2(t+\tau), \quad 0 < t < T_2 - \tau. \tag{38}$$

Moreover, conditions (27), (29)–(31) will take the forms

$$\left.\begin{array}{c} y_1(t) = \varphi(t),\ t \in [-\tau, 0]; \quad y_2(t) = y_1(t + T_1),\ t \in (-\tau, 0); \\ y_2(0) = y_1(T_1); \quad y_2(t) = 0,\ t \in [T_2 - \tau, T_2], \end{array}\right\} \tag{39}$$

respectively, while the Kirchhoff condition (33) can be represented as

$$y_1'(T_1) + (b_1 - b_2)y_1(T_1) + (c_1 - c_2)y_1(T_1 - \tau) = y_2'(0). \tag{40}$$

In particular, if $b_1 = b_2$, $c_1 = c_2$, the problem (37)–(40) takes the form (21), (22), (24), where $a = 0$, $b = b_1$, $c = c_1$, $T = T_1 + T_2$, while

$$y(t) := y_1(t),\ \ 0 \leq t \leq T_1, \quad y(t) := y_2(t - T_1),\ \ t > T_1. \tag{41}$$

**Conclusion**: Artificial reproducing copies of one and the same scenario starting from a certain point of the interval and employing appropriate weights in the corresponding energy functional leads to the same optimal control as in the original interval case.

**Example II.** Consider the simple control problem

$$y'(t) = u(t), \quad y(0) = 2, \quad y(T) = 0. \tag{42}$$

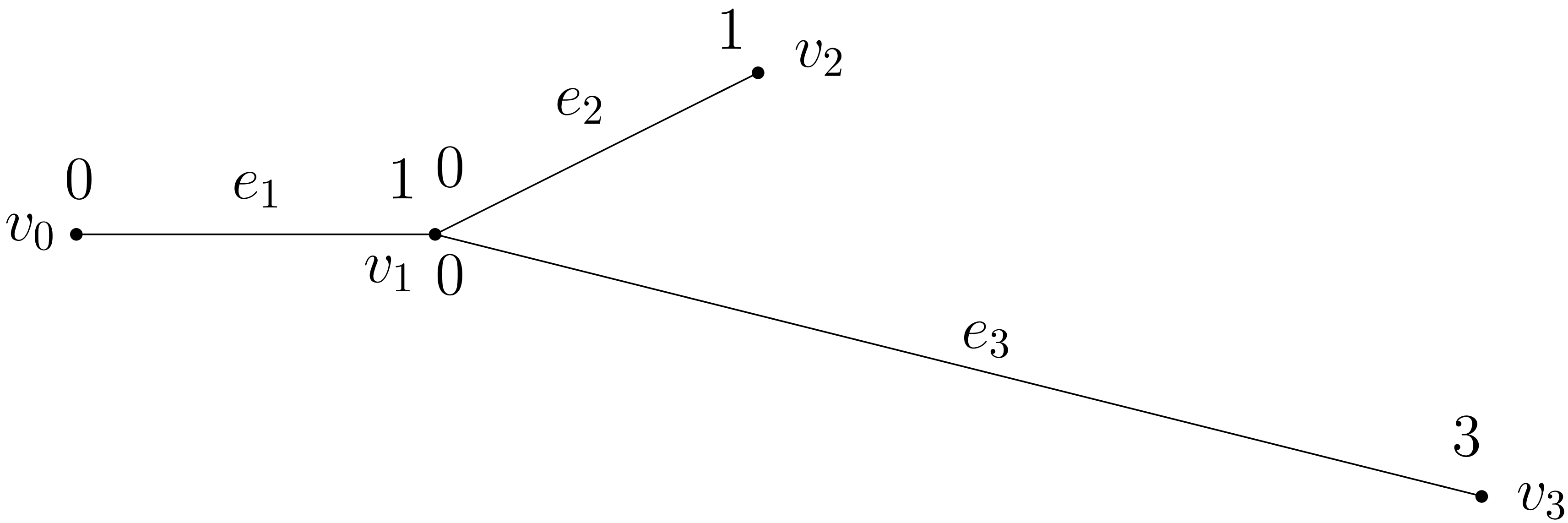
According to Theorems 1 and 2, the optimal control $u(t) \in L_2(0, T)$ is unique, while the corresponding optimal trajectory $y(t)$ obeys the BVP

$$y''(t) = 0, \quad y(0) = 2, \quad y(T) = 0. \tag{43}$$

Assume that we need to find the optimal control but we know only that $T$ may be equal either to 2 or to 4. The precise information on $T$ will be available only starting from $t = 1$.

**Main question:** Which control $u(t)$ should be chosen before the time point $t = 1$?

An **answer** comes when we extend the system (42) to a 3-star graph $\Gamma_3$ taking into account both possibilities.




Denote by $y_1(t)$ a trajectory on the interval $(0,1)$, which can be only common, and by $\tilde{y}_2(t)$ and $\tilde{y}_3(t)$ – two possible trajectories on $(1,2)$ and $(1,4)$, respectively. Then we arrive at the control problem on $\Gamma_3$ :

$$y_j'(t) = u_j(t), \quad 0 < t < 1, \quad j = 1,2, \qquad y_3'(t) = u_3(t), \quad 0 < t < 3,$$

$$y_1(0) = 2, \quad y_1(1) = y_2(0) = y_3(0), \quad y_2(1) = y_3(3) = 0,$$

where $y_j(t) = \tilde{y}_j(t+1)$ for $j = 2,3$.

Assume that the probability of $T = 2$ equals $p \in (0, 1)$. Hence, $1 - p$ is the probability of $T = 4$. Thus, in the corresponding energy functional, we have $\alpha_1 = 1$, $\alpha_2 = p$ and $\alpha_3 = 1 - p$, while $T_1 = T_2 = 1$ and $T_3 = 3$.

According to Th. 3, the optimal trajectory $[y_1, y_2, y_3]$ solves the BVP

$$y_j''(t) = 0, \quad 0 < t < 1, \quad j = 1, 2, \qquad y_3''(t) = 0, \quad 0 < t < 3, \tag{44}$$

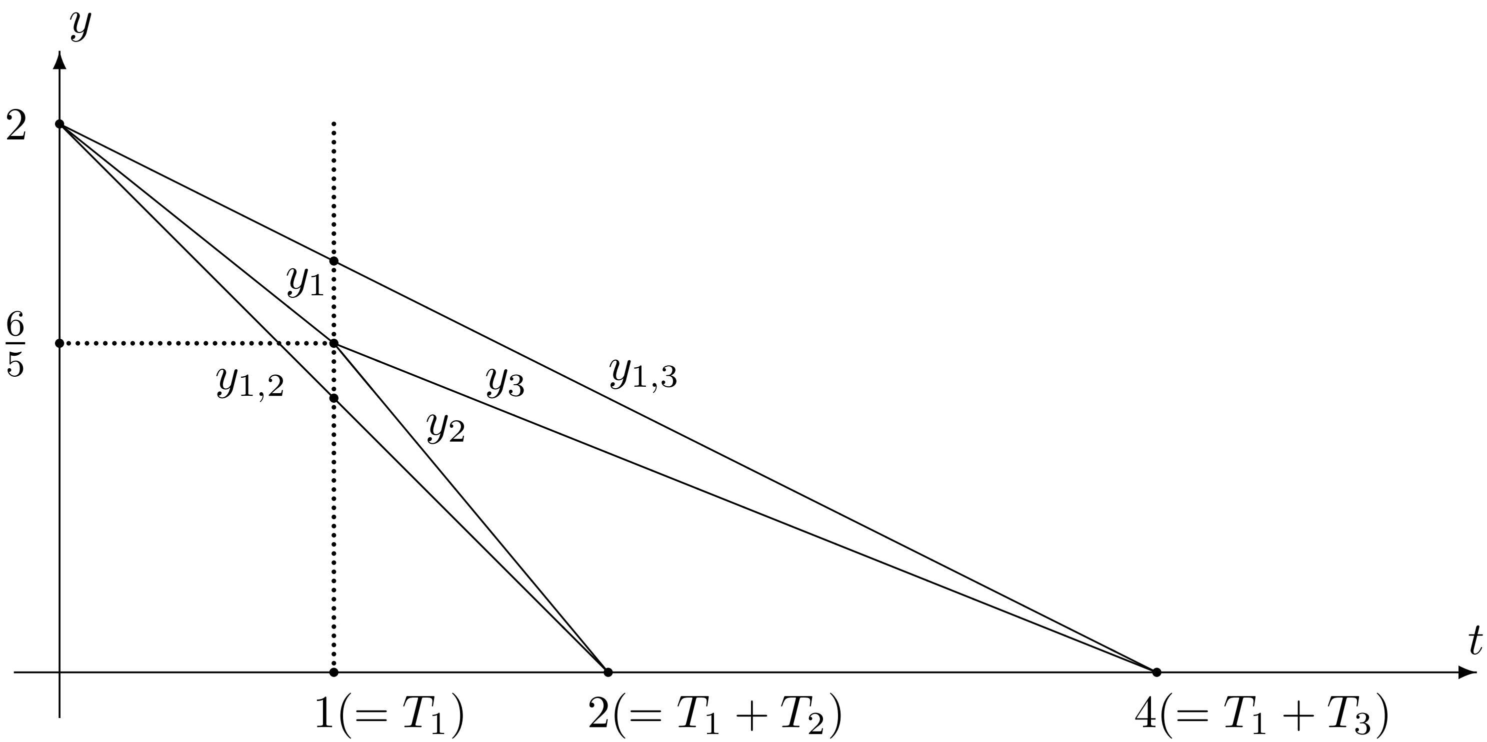
$$y_1(1) = y_2(0) = y_3(0), \quad y_1'(1) = p y_2'(0) + (1 - p) y_3'(0), \tag{45}$$

$$y_1(0) = 2, \quad y_2(1) = y_3(3) = 0, \tag{46}$$

whose solution has the form

$$y_1(t) = 2(1 - t) + \frac{3t}{p + 2}, \quad y_2(t) = 3\frac{1 - t}{p + 2}, \quad 0 < t < 1, \tag{47}$$

$$y_3(t) = \frac{3 - t}{p + 2}, \quad 0 < t < 3. \tag{48}$$




Although the trajectory $y_{1,2}$ is shorter than the composite trajectory $[y_1, y_2]$, while $y_{1,3}$ is shorter than $[y_1, y_3]$, these composite trajectories allow one to take more advantageous position at the time point $t = T_1$.

Finally, note the point $y_1(1) = \frac{3}{p+2}$ sweeps the interval $(y_{1,2}(1), y_{1,3}(1)) = (1, \frac{3}{2})$ as soon as $p$ ranges over $(0, 1)$.

[22] Buterin S.A. *On damping a control system of arbitrary order with global aftereffect on a tree*, Math. Notes 115 (2024) no. 6, 877–896.

Consider the equations of neutral type

$$\ell y(t) := \sum_{k=0}^{n} \Big( b_k(t) y^{(k)}(t) + c_k(t) y^{(k)}(t-\tau) \Big) = u(t), \quad t > 0, \tag{49}$$

with a constant delay $\tau > 0$ and complex-valued components such that

$$\forall\, T > 0 \quad b_n,\, b_n^{-1},\, c_n \in L_\infty(0,T), \quad u,\, b_k,\, c_k \in L_2(0,T), \quad k = \overline{0, n-1}.$$

Prehistory of the system is determined by the conditions:

$$y(t) = \varphi(t) \in W_2^n[-\tau, 0], \quad t \in (-\tau, 0), \tag{50}$$

$$y^{(k)}(0) = \varphi^{(k)}(0), \quad k = \overline{0, n-1}. \tag{51}$$

Fix $T > 2\tau$ and find $u(t) \in L_2(0,T)$ bringing into the equilibrium state

$$y(t) = 0, \quad t \in [T-\tau, T]. \tag{52}$$

Thus, we arrive at the variational problem

$$J(y) = \int_0^T |\ell y(t)|^2 \, dt \to \min \tag{53}$$

under the conditions (50)–(52).

Put

$$\tilde{\ell}_k y(t) := \overline{b_k(t)} \ell y(t) + \overline{c_k(t+\tau)} \ell y(t+\tau), \quad 0 < t < T - \tau, \quad k = \overline{0, n},$$

and introduce the quasi-derivatives

$$\left.\begin{aligned} y^{\langle n \rangle}(t) &:= \tilde{\ell}_n y(t), \\ y^{\langle n+l \rangle}(t) &:= \tilde{\ell}_{n-l} y(t) - (y^{\langle n+l-1 \rangle})'(t), \quad l = \overline{1, n}. \end{aligned}\right\} \tag{54}$$

**Theorem 5.** *A function* $y(t) \in W_2^n[-\tau, T]$ *is a solution of the variational problem (50)–(53) if and only if it obeys the conditions*

$$y^{\langle k \rangle}(t) \in W_1^1[0, T - \tau], \quad k = \overline{n, 2n-1}, \tag{55}$$

*and solves the self-adjoint boundary value problem* $\mathcal{B}$ *for the equation*

$$y^{\langle 2n \rangle}(t) = 0, \quad 0 < t < T - \tau, \tag{56}$$

*under the conditions (50)–(52).*

**Theorem 6.** *The problem* $\mathcal{B}$ *has a unique solution* $y(t) \in W_2^n[-\tau, T]$ *obeying (55). Moreover, the estimate*

$$\|y\|_{W_2^n[-\tau,T]} \leq C\|\varphi\|_{W_2^n[-\tau,0]},$$

*is fulfilled, where* $C$ *is independent of* $\varphi(t)$*.*

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**[30]** Bondarenko N.P. *Linear differential operators with distribution coefficients of various singularity orders*, Math. Meth. Appl. Sci. 46 (2022) no.6, 6639–6659.

[28] Mirzoev K.A. and Shkalikov A.A. *Differential operators of even order with distribution coefficients*, Math. Notes 99 (2016) no. 5, 779–784.

$$\sum_{k,s=0}^{n} (r_{ks}(t)y^{(n-k)})^{(n-s)}, \quad -\infty \le a < t < b \le \infty, \tag{57}$$

where $r_{ks}$ are complex-valued functions such that

$$\frac{1}{\sqrt{|r_{00}|}}, \ \frac{1}{\sqrt{|r_{00}|}} R_{ks} \ \in \ L_{2,\mathrm{loc}}(a,b), \quad k, s = \overline{0, n}, \tag{58}$$

while

$$R_{ks}^{(l)} = r_{s,k}, \quad l = \min\{k, s\}. \tag{59}$$

**[26]** Neiman-Zade M.I. and Shkalikov A.A. *Schrödinger operators with singular potentials from spaces of multipliers*, Math. Notes 66 (1999) no. 5, 599–607.

**[27]** Savchuk A.M. and Shkalikov A.A. *Sturm–Liouville operators with singular potentials*, Math. Notes 66 (1999) no. 6, 741–753.

Consider the differential expression

$$\ell_0 y(t) := -y''(t) + q(t)y(t), \quad 0 < t < 1, \tag{60}$$

where $q(t)$ is a complex distribution in $W_2^{-1}[0,1]$. The latter means

$$q = \sigma', \quad \sigma \in L_2(0,1). \tag{61}$$

Consider the quasi-derivative

$$y^{[1]} = y' - \sigma y. \tag{62}$$

Then

$$\ell_0 y = -(y^{[1]})' - \sigma y^{[1]} - \sigma^2 y =: y^{[2]}. \tag{63}$$

## Example 1

Let $n = 1$, $b_1 = 1$, $c_0 = c_1 = 0$, while $b_0 \in L_2(0,1)$ is real valued. Then

$$\ell y = y' + b_0 y, \tag{64}$$

$$\tilde{\ell}_0 y = b_0 \ell y = b_0 y' + b_0^2 y, \quad \tilde{\ell}_1 y = \ell y, \tag{65}$$

$$y^{\langle 1 \rangle} = \tilde{\ell}_1 y = y' + b_0 y, \tag{66}$$

$$y^{\langle 2 \rangle} = \tilde{\ell}_0 y - (y^{\langle 1 \rangle})' = -(y^{\langle 1 \rangle})' + b_0 y^{\langle 1 \rangle} \tag{67}$$

or

$$y^{\langle 2 \rangle} = -y'' + qy, \tag{68}$$

where $q \in W_2^{-1}[0,1]$. More precisely,

$$q = -b_0' + b_0^2, \quad b_0 \in L_2(0,1). \tag{69}$$

Substituting $b_0 = -u'/u$ into (69) gives

$$-u'' + qu = 0. \tag{70}$$

**Example 2**

Let

$$n = 1, \quad b_1 = 1, \quad c_1 = a, \quad b_0 = b, \quad c_0 = c, \tag{71}$$

where $a, b, c \in \mathbb{R}$. Then

$$\ell y(t) = y'(t) + ay'(t-\tau) + by(t) + cy(t-\tau) \tag{72}$$

and

$$y^{\langle 1\rangle}(t) = \ell y(t) + a\ell y(t+\tau), \quad t \in [0, T-\tau]. \tag{73}$$

More precisely,

$$\begin{aligned} y^{\langle 1\rangle}(t) = (1+a^2)y'(t) + ay'(t-\tau) + ay'(t+\tau) \\ +(ac+b)y(t) + cy(t-\tau) + aby(t+\tau). \end{aligned} \tag{74}$$

**[18]** Skubachevskii A.L. *On the problem of damping a control system with aftereffect*, Russian Acad. Sci. Dokl. Math. 49 (1994) 282–286.

## Example 3

Let $n = 2$, $b_2 = c_1 = 1$, $b_0 = b_1 = c_0 = c_2 = 0$.

Then we have an expression of the retarded type:

$$\ell y(t) = y''(t) + y'(t - \tau). \tag{75}$$

Hence, for $t \in [0, T - \tau]$, we have

$$\tilde{\ell}_0 y(t) = 0, \quad \tilde{\ell}_1 y(t) = \ell y(t + \tau), \quad \tilde{\ell}_2 y(t) = \ell y(t), \tag{76}$$

$$y^{\langle 2 \rangle}(t) = \tilde{\ell}_2 y(t) = y''(t) + y'(t - \tau), \tag{77}$$

$$\begin{aligned} y^{\langle 3 \rangle}(t) &= \tilde{\ell}_1 y(t) - (y^{\langle 2 \rangle})'(t) \\ &= -y'''(t) + y''(t + \tau) - y''(t - \tau) + y'(t). \end{aligned} \tag{78}$$

Thus, assuming that $T = 4\tau$ and $\varphi(t) \in W_2^2[-\tau, 0] \setminus W_1^3[-\tau, 0]$, we have

$$y'''(t) \in W_1^1[0, 3\tau] \quad \Rightarrow \quad y^{\langle 3 \rangle}(t) \notin W_1^1[0, 3\tau], \tag{79}$$

which holds also for $\varphi(t) \in W_1^3[-\tau, 0]$ if $y''(0) \neq \varphi''(0)$.

Let $\mathcal{T}$ be a tree with vertices $\{v_0, v_1, \dots, v_m\}$ and edges $\{e_1, \dots, e_m\}$; $\{v_0, v_{d+1}, \dots, v_m\}$ are boundary vertices and $\{v_1, \dots, v_d\}$ are internal.

Let $e_j = [v_{k_j}, v_j], j = \overline{1, m}$, and $k_1 = 0$. The vertex $v_0$ is labelled as *root*.

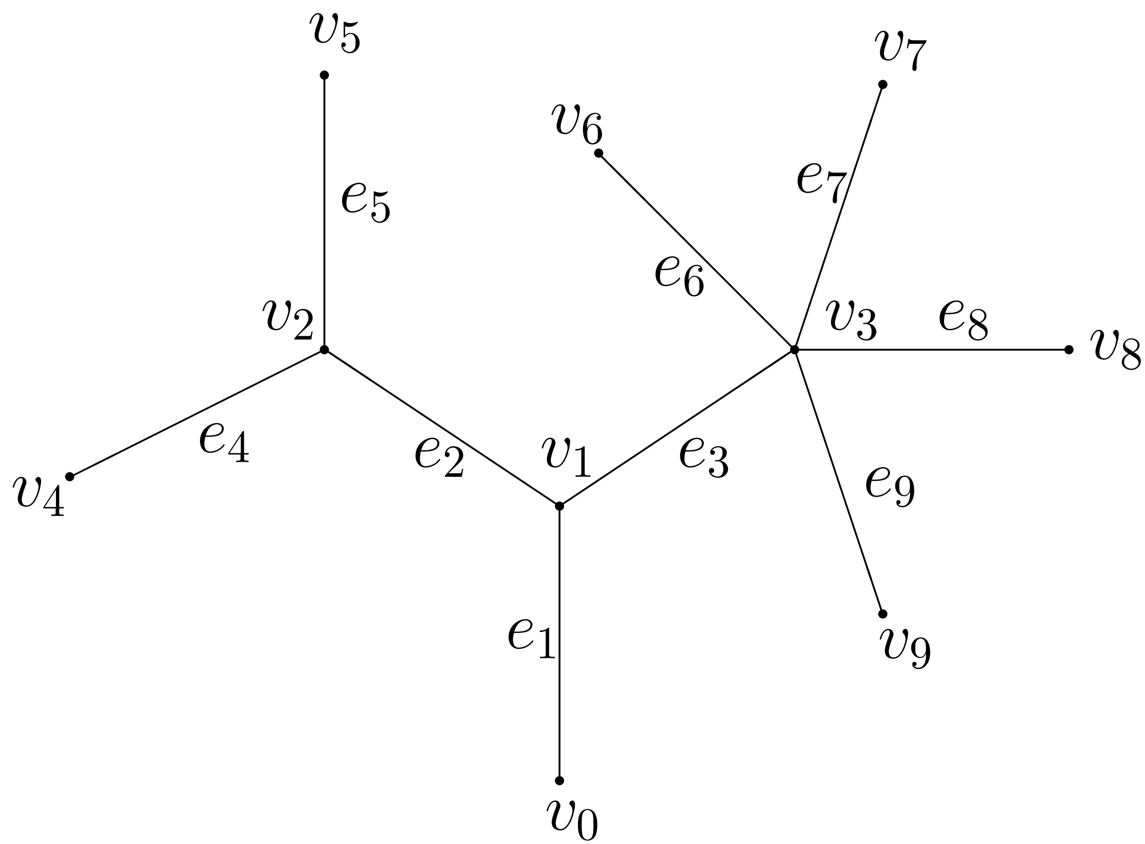


Fig. 2. A non-star tree with $m = 9$ and $d = 3$,

$$k_1 = 0, \quad k_2 = k_3 = 1, \quad k_4 = k_5 = 2, \quad k_6 = k_7 = k_8 = k_9 = 3.$$

The mapping $k_j : \{1, \ldots, m\} \to \{0, 1, \ldots, d\}$ uniquely determines the structure of $\mathcal{T}$.

Indeed, denote

$$V_j := \{\nu : k_\nu = j\}.$$

Then for each $j = \overline{0, d}$, the set $\{e_\nu\}_{\nu \in V_j}$ coincides with the set of edges emanating from the vertex $v_j$.

By a function $y$ on $\mathcal{T}$ we mean an $m$-tuple $y = [y_1, \ldots, y_m]$ whose component $y_j$ is defined on the edge $e_j$, i.e. $y_j = y_j(t)$, $t \in [0, T_j]$, where $T_j$ is the length of $e_j$.

We also say that the function $y$ is defined on the extended tree $\mathcal{T}_\tau$ if it is defined on $\mathcal{T}$ and its first component $y_1(t)$ is defined also for $t \in [-\tau, 0)$.

For definiteness, let $\tau < T_j$ for all $j = \overline{1,m}$.

Consider the control system determined by the Cauchy problem on $\mathcal{T}_\tau$ :

$$\ell_j y(t) := \sum_{k=0}^{n} \Big( b_{k,j}(t) y_j^{(k)}(t) + c_{k,j}(t) y_j^{(k)}(t-\tau) \Big) = u_j(t), \qquad 0 < t < T_j, \quad j = \overline{1,m}, \tag{80}$$

$$y_j(t) = y_{k_j}(t + T_{k_j}), \quad t \in (-\tau, 0), \quad j = \overline{2,m}, \tag{81}$$

$$y_j^{(k)}(0) = y_{k_j}^{(k)}(T_{k_j}), \quad k = \overline{0, n-1}, \quad j = \overline{2,m}, \tag{82}$$

$$y_1(t) = \varphi(t) \in W_2^n[-\tau, 0], \quad t \in (-\tau, 0), \tag{83}$$

$$y_1^{(k)}(0) = \varphi^{(k)}(0), \quad k = \overline{0, n-1}, \tag{84}$$

with complex-valued $\varphi(t)$ and

$$b_{n,j}, \frac{1}{b_{n,j}}, c_{n,j} \in L_\infty(0, T_j), \quad u_j, b_{k,j}, c_{k,j} \in L_2(0, T_j), \;\; k = \overline{0, n-1}.$$

The Cauchy problem (80)–(84) has a unique solution

$$y = [y_1, \dots, y_m] \in W_2^n(\mathcal{T}_\tau) := W_2^n[-\tau, T_1] \oplus \bigoplus_{j=2}^{m} W_2^n[0, T_j].$$

One needs to find a control

$$u = [u_1, \dots, u_m] \in L_2(\mathcal{T}) := \bigoplus_{j=1}^{m} L_2(0, T_j)$$

bringing the system (80)–(84) into the equilibrium state

$$y_j(t) = 0, \quad t \in [T_j - \tau, T_j], \quad j = \overline{d+1, m}, \tag{85}$$

and minimizing the norm $\|u\|_{L_2(\mathcal{T})} = \sqrt{\sum_{j=1}^{m} \|u_j\|^2_{L_2(0,T_j)}}$.

Thus, we arrive at the variational problem

$$\mathcal{J}(y) := \sum_{j=1}^{m} \int_0^{T_j} |\ell_j y(t)|^2 \, dt \to \min \tag{86}$$

for the functions $y = [y_1, \dots, y_m]$ on $\mathcal{T}_\tau$ obeying (81)–(85).

Denote

$$\ell_{k,j}y(t) = \overline{b_{k,j}(t)}\ell_j y(t) + \begin{cases} \overline{c_{k,j}(t+\tau)}\ell_j y(t+\tau), \\ \qquad 0 < t < T_j - \tau, \quad j = \overline{1,m}, \\ \\ \sum\limits_{\nu \in V_j} \overline{c_{k,\nu}(t+\tau-T_j)}\ell_\nu y(t+\tau-T_j), \\ \qquad T_j - \tau < t < T_j, \quad j = \overline{1,d}. \end{cases} \tag{87}$$

Consider the quasi-derivatives

$$\left. \begin{array}{c} y_j^{\langle n \rangle}(t) := \ell_{n,j} y(t), \\ \\ y_j^{\langle n+l \rangle}(t) := \ell_{n-l,j} y(t) - (y_j^{\langle n+l-1 \rangle})'(t), \ \ l = \overline{1,n}, \end{array} \right\} \quad j = \overline{1,m}. \tag{88}$$

On the set of functions $y = [y_1, \dots, y_m] \in W_2^n(\mathcal{T}_\tau)$ obeying the conditions

$$y_j^{\langle k \rangle}(t) \in W_1^1[0, l_j], \quad k = \overline{n, 2n-1}, \quad j = \overline{1, m}, \tag{89}$$

we consider the boundary value problem $\mathcal{B}$ :

$$y_j^{\langle 2n \rangle}(t) = 0, \quad 0 < t < l_j, \quad j = \overline{1, m}, \tag{90}$$

$$y_j(t) = y_{k_j}(t + T_{k_j}), \quad t \in (-\tau, 0), \quad j = \overline{2, m}, \tag{81}$$

$$y_j^{(k)}(0) = y_{k_j}^{(k)}(T_{k_j}), \quad k = \overline{0, n-1}, \quad j = \overline{2, m}, \tag{82}$$

$$y_1(t) = \varphi(t), \quad t \in (-\tau, 0), \tag{83}$$

$$y_1^{(k)}(0) = \varphi^{(k)}(0), \quad k = \overline{0, n-1}, \tag{84}$$

$$y_j(t) = 0, \quad t \in [T_j - \tau, T_j], \quad j = \overline{d+1, m}, \tag{85}$$

$$y_j^{\langle k \rangle}(l_j) = \sum_{\nu \in V_j} y_\nu^{\langle k \rangle}(0), \quad j = \overline{1, d}, \quad k = \overline{n, 2n-1}. \tag{91}$$

**Theorem 7.** *The function $y \in W_2^n(\mathcal{T}_\tau)$ is a solution of the variational problem (81)–(86) if and only if it obeys the conditions*

$$y_j^{\langle k\rangle}(t) \in W_1^1[0, l_j], \quad k = \overline{n, 2n-1}, \quad j = \overline{1, m}, \tag{89}$$

*and solves the boundary value problem $\mathcal{B}$.*

**Theorem 8.** *The problem $\mathcal{B}$ has a unique solution $y \in W_2^n(\mathcal{T}_\tau)$, obeying the conditions (89). Moreover, the estimate*

$$\|y\|_{W_2^n(\mathcal{T}_\tau)} \leq C\|\varphi\|_{W_2^n[-\tau,0]}, \tag{92}$$

*holds, where $C$ does not depend on $\varphi(t)$.*

**Thank you for your attention!**