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Inverse problems for the fourth-order differential operators

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Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in (0,1), \tag{1}$$

$q \in L_1[0,1]$ is a real-valued potential.

$$\{\lambda_n\}_{n\geqslant 1}: \quad y(0) = 0, \quad y(1) = 0,$$
$$\{\mu_n\}_{n\geqslant 1}: \quad y'(0) = 0, \quad y(1) = 0.$$

Theorem (G. Borg, 1946)

The two spectra $\{\lambda_n\}_{n\geqslant 1}$ and $\{\mu_n\}_{n\geqslant 1}$ uniquely specify the potential q.

Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in (0,1), \tag{1}$$
$$y(0) = y(1) = 0. \tag{2}$$

- $\{\lambda_n\}_{n\geqslant 1}$ — eigenvalues of (1)–(2),
- $\{y_n\}_{n\geqslant 1}$ — normalized eigenfunctions: $\int_0^1 y_n^2(x)\,dx = 1$,
- $\alpha_n := y_n'(0) > 0$ — norming constants.

Theorem (V.A. Marchenko, 1950)

The spectral data $\{\lambda_n, \alpha_n\}_{n\geqslant 1}$ uniquely specify q.

- Gel'fand, I.M.; Levitan, B.M. On the determination of a differential equation from its spectral function, Izv. Akad. Nauk SSSR, Ser. Mat. (1951).

Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in (0,1). \tag{1}$$

Denote by $S(x,\lambda)$ and $C(x,\lambda)$ the solutions of (1) satisfying

$$S(0,\lambda) = 0, \quad S'(0,\lambda) = 1, \quad C(0,\lambda) = 1, \quad C'(0,\lambda) = 0.$$

- $S(x,\lambda)$ and $C(x,\lambda)$ are entire analytic in λ for each fixed $x \in [0,1]$.
- $\{\lambda_n\}_{n\geqslant 1}$ are the zeros of $S(1,\lambda)$.
- $\{\mu_n\}_{n\geqslant 1}$ are the zeros of $C(1,\lambda)$.
- Weyl function $M(\lambda) := \dfrac{C(1,\lambda)}{S(1,\lambda)}$ is meromorphic.
- $\operatorname*{Res}_{\lambda=\lambda_n} M(\lambda) = \alpha_n^2$.
- The three sets of the spectral data: $\{\lambda_n, \mu_n\}_{n\geqslant 1}$, $\{\lambda_n, \alpha_n\}_{n\geqslant 1}$, and $M(\lambda)$ uniquely specify each other and the potential q.

Barcilon's problem

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y, \quad x \in (0,1), \tag{3}$$
$$p, q \in L_1[0,1].$$

$$\begin{aligned} \mathfrak{S}_{12}: &\quad y(0) = y'(0) = 0, \quad y(1) = y'(1) = 0, \\ \mathfrak{S}_{13}: &\quad y(0) = y''(0) = 0, \quad y(1) = y'(1) = 0, \\ \mathfrak{S}_{23}: &\quad y'(0) = y''(0) = 0, \quad y(1) = y'(1) = 0. \end{aligned}$$

Inverse problem

Given the three spectra $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, and $\mathfrak{S}_{23}$, find p and q.

1. Barcilon V. On the uniqueness of inverse eigenvalue problems, Geophysical Journal International 38 (1974), no. 2, 287–298.
2. Barcilon V. On the solution of inverse eigenvalue problems of high orders, Geophysical Journal International 39 (1974), no. 1, 143–154.

Uniqueness was not rigorously proved.

McLaughlin's problem

- McLaughlin, J.R. Higher order inverse eigenvalue problems. In: Everitt, W., Sleeman, B. (eds) Ordinary and Partial Differential Equations. Lecture Notes in Mathematics, vol 964. Springer, Berlin, Heidelberg, 1982.

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y, \quad x \in (0,1), \tag{3}$$

$$\left.\begin{array}{c} U_1(y) := y''(0) + ay'(0) - by(0) = 0, \\ U_2(y) := y^{[3]}(0) + by'(0) + cy(0) = 0, \\ y(1) = y'(1) = 0, \end{array}\right\} \tag{4}$$

$y^{[3]} := y''' - py'$ — quasi-derivative.

- $\{\lambda_n\}_{n \geqslant 1}$ — eigenvalues, assume that they are simple.
- $\{y_n(x)\}_{n \geqslant 1}$ — eigenfunctions, $\int_0^1 y_n^2(x)\,dx = 1$, $n \geqslant 1$.
- $\gamma_n := y_n(0)$, $\xi_n := y_n'(0)$ — norming constants.

Inverse problem

Given the spectral data $\{\lambda_n, \gamma_n, \xi_n\}_{n \geqslant 1}$, find p, q, a, b, and c.

J.R. McLaughlin studied solvability for the inverse problem under a restrictive condition that transformation operator exists. Uniqueness was an open question.

Transformation operators

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y, \quad x \in (0, 1), \tag{3}$$

- $y(x, \lambda)$ — solution of (3) with $p(x)$ and $q(x)$.
- $y_0(x, \lambda)$ — solution of (3) with $p = q = 0$.

$$y(x, \lambda) = y_0(x, \lambda) + \int_0^x K(x, t) y_0(t, \lambda)\, dt.$$

Transformation operators are effective for order 2, ineffective for higher orders.

Transformation operators & inverse spectral problems for higher-orders:
L.A. Sakhnovich, I.G. Khachatryan, M.M. Malamud, ...
(analytic / piecewise analytic coefficients)

Inverse problems for higher orders: general approach

$$y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)}, \quad n > 2. \tag{5}$$

The theory of inverse spectral problems for higher-order differential operators with regular coefficients $p_k \in W_1^k[0,1]$ has been created by V.A. Yurko.

N.P. Bondarenko has transferred those results to differential operators with distribution coefficients.

1. Yurko, V.A. Reconstruction of higher-order differential operators, Differ. Equ. (1989).
2. Yurko, V.A. Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-Posed Problems Series, VNU Science, Utrecht (2002).
3. Bondarenko, N.P. Linear differential operators with distribution coefficients of various singularity orders, Math. Meth. Appl. Sci. (2023).

Weyl-Yurko matrix

$$y^{(4)} - (p(x)y')' - (r(x)y)' - r(x)y' + q(x)y = \lambda y, \quad x \in (0,1), \tag{6}$$

where $p, q, r \in L_1[0,1]$. Quasi-derivatives:

$$y^{[j]} := y^{(j)}, \quad j = 0,1,2, \qquad y^{[3]} := y''' - py' - ry.$$

Linear forms:

$$U_s(y) := y^{[s-1]}(0), \quad s = \overline{1,4}, \qquad V_j(y) := y^{[j-1]}(1), \quad j = \overline{1,4}. \tag{7}$$

Denote by $\{\Phi_k(x,\lambda)\}_{k=1}^4$ the solutions of (6) satisfying

$$U_s(\Phi_k) = \delta_{sk}, \quad s = \overline{1,k}, \qquad V_j(\Phi_k) = 0, \quad j = \overline{1,4-k}. \tag{8}$$

where δ_{sk} is the Kronecker delta. $\{\Phi_k(x,\lambda)\}_{k=1}^4$ are called the Weyl solutions, they are meromorphic in λ for each fixed $x \in [0,1]$.

Weyl-Yurko matrix $M(\lambda) := [U_s(\Phi_k)]_{s,k=1}^4$.

Theorem 1 (Yurko — smooth, Bond. — non-smooth)

The Weyl-Yurko matrix $M(\lambda)$ uniquely specifies p, q, and r.

Weyl-Yurko matrix

$$M(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21}(\lambda) & 1 & 0 & 0 \\ m_{31}(\lambda) & m_{32}(\lambda) & 1 & 0 \\ m_{41}(\lambda) & m_{42}(\lambda) & m_{43}(\lambda) & 1 \end{bmatrix},$$

Denote by $\{C_k(x,\lambda)\}_{k=1}^4$ the solutions of equation (6) satisfying

$$U_s(C_k) = \delta_{sk}, \quad s = \overline{1,4}. \tag{9}$$

Then

$$m_{jk}(\lambda) = -\frac{\Delta_{jk}(\lambda)}{\Delta_{kk}(\lambda)}, \quad 1 \leqslant k < j \leqslant 4, \tag{10}$$

where $\Delta_{kk}(\lambda) := \det\left([V_{5-p}(C_r)]_{s,p=k+1}^4\right)$ and $\Delta_{jk}(\lambda)$ is obtained from $\Delta_{kk}(\lambda)$ by replacing C_j by C_k.

The zeros of $\Delta_{jk}(\lambda)$ coincide with the eigenvalues of the boundary value problem $\mathcal{L}_{jk}$ for equation (6) with the boundary conditions:

$$U_\xi(y) = 0, \quad \xi = \overline{1,k-1}, j, \quad V_\eta(y) = 0, \quad \eta = \overline{1,4-k}. \tag{11}$$

Put $\mathcal{L}_k := \mathcal{L}_{kk}$, $k = 1,2,3$.

Separation condition

Separation condition

For $k = 1, 2$, the problems $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$ have no common eigenvalues.

Theorem 2 (Yurko — smooth, Bond. — non-smooth)

Under the separation condition, the functions $m_{21}(\lambda)$, $m_{32}(\lambda)$, and $m_{43}(\lambda)$ uniquely specify p, q, and r.

$$M(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21}(\lambda) & 1 & 0 & 0 \\ m_{31}(\lambda) & m_{32}(\lambda) & 1 & 0 \\ m_{41}(\lambda) & m_{42}(\lambda) & m_{43}(\lambda) & 1 \end{bmatrix}.$$

- Lejbenzon, Z.L. The uniqueness of the solution of the inverse problem for ordinary differential operators of order $n \geqslant 2$ and the transformation of such operators, Sov. Math. Dokl. (1962).

Separation condition

$$y^{(4)} - (p(x)y')' - (r(x)y)' - r(x)y' + q(x)y = \lambda y, \quad x \in (0, 1), \qquad (6)$$

FAQ: Do there exist such p, q, r that the separation condition holds?

1. Separation condition holds for $p = q = r = 0$.
2. Recently, the spectral data characterization was obtained in [Bondarenko N.P., Mathematics, 2024] for the class of equations (6) with real-valued $p \in W_2^1[0,1]$, $ir \in L_2[0,1]$, $q \in W_2^{-1}[0,1]$, and the eigenvalues of the problems $\mathcal{L}_k$, $k = 1, 2, 3$, being simple and satisfying the separation condition.
3. One can achieve the separation condition by a finite perturbation of the spectral data for any p, q, r.

Barcilon's problem

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y, \quad x \in (0,1), \tag{3}$$

Theorem 3 (Guan, Yang & Bond., 2023)

Under the separation condition, the spectra $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, and $\mathfrak{S}_{23}$ uniquely specify the coefficients $p, q \in L_1[0,1]$ of equation (3).

Lemma 1

The spectra $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, $\mathfrak{S}_{23}$ coincide with the zeros of the functions $\Delta_{22}(\lambda)$, $\Delta_{32}(\lambda)$, $\Delta_{42}(\lambda)$.

Hadamard's Factorization Theorem implies

$$\Delta_{j2}(\lambda) = c_j \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\mu_{jn}}\right), \quad j = 2, 3, 4. \tag{12}$$

$$\mathfrak{S}_{12}, \mathfrak{S}_{13}, \mathfrak{S}_{23} \;\Rightarrow\; \Delta_{22}(\lambda), \Delta_{32}(\lambda), \Delta_{42}(\lambda) \;\Rightarrow\; m_{32}(\lambda), m_{42}(\lambda).$$

Barcilon's problem

$$M(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21}(\lambda) & 1 & 0 & 0 \\ m_{31}(\lambda) & m_{32}(\lambda) & 1 & 0 \\ m_{41}(\lambda) & m_{42}(\lambda) & m_{43}(\lambda) & 1 \end{bmatrix}.$$

Lemma 2

For the Weyl-Yurko matrix of equation (3), the following relations hold:

$$m_{43}(\lambda) = m_{21}(\lambda), \tag{13}$$

$$m_{42}(\lambda) - m_{32}(\lambda)m_{21}(\lambda) + m_{31}(\lambda) = 0. \tag{14}$$

For Lemma 2, the equality $r = 0$ and special structure of the boundary conditions are crucial.

Denote $\mathfrak{S}_{12} = \{\lambda_n\}_{n \geqslant 1}$. Using (14), one can find $\{m_{21}(\lambda_n)\}_{n \geqslant 1}$ from $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, $\mathfrak{S}_{23}$.

Lemma 3

The values $\{m_{21}(\lambda_n)\}_{n \geqslant 1}$ uniquely specify $m_{21}(\lambda)$.

Relationship between Barcilon's and McLaughlin's problems

$$y^{(4)} - (p(x)y')' + q(x)y = \lambda y, \quad x \in (0,1), \tag{3}$$

$$\left.\begin{array}{c} U_1(y) := y''(0) + ay'(0) - by(0) = 0, \\ U_2(y) := y^{[3]}(0) + by'(0) + cy(0) = 0, \\ y(1) = y'(1) = 0. \end{array}\right\} \tag{4}$$

Introduce the linear forms

$$U_3(y) = y(0), \quad U_4(y) = y'(0), \quad V_s(y) = y^{[s-1]}(1), \quad s = \overline{1,4}. \tag{15}$$

Let $\mathfrak{S}_{jk}$ for $(j,k) \in \{(1,2),(1,3),(2,3)\}$ be the spectra for (3) with the boundary conditions

$$U_j(y) = U_k(y) = 0, \quad V_1(y) = V_2(y) = 0. \tag{16}$$

Recall that $\mathfrak{S}_{12} = \{\lambda_n\}_{n \geqslant 1}$ are assumed to be simple, $\gamma_n := y_n(0)$, $\xi_n := y_n'(0)$, $y_n(x)$ — normalized eigenfunctions: $\int_0^1 y_n^2(x)\,dx = 1$.

Theorem 4 (Bond., 2023)

Under the separation condition, the three spectra $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, $\mathfrak{S}_{23}$ uniquely determine McLaughlin's data $\{\lambda_n, \gamma_n, \xi_n\}_{n \geqslant 1}$ (up to the signs of γ_n and ξ_n) and vice versa.

Relationship between Barcilon's and McLaughlin's problems

$$\mathfrak{S}_{12}, \mathfrak{S}_{13}, \mathfrak{S}_{23} \;\Leftrightarrow\; \Delta_{22}(\lambda),\, \Delta_{32}(\lambda),\, \Delta_{42}(\lambda) \;\Leftrightarrow\; \{\lambda_n, \Delta_{32}(\lambda_n), \Delta_{42}(\lambda_n)\}_{n\geqslant 1}$$

$$\Delta_{32}(\lambda_n) = \tfrac{d}{d\lambda}\Delta_{22}(\lambda_n)\gamma_n^2, \quad \Delta_{42}(\lambda_n) = \tfrac{d}{d\lambda}\Delta_{22}(\lambda_n)\xi_n\gamma_n. \tag{17}$$

Theorem 4 implies

Theorem 5 (Bond., 2023)

Under the separation condition, the spectral data $\{\lambda_n, \gamma_n, \xi_n\}_{n\geqslant 1}$ *uniquely specify* p*,* q*,* a*,* b*, and* c*.*

Let us show that the separation condition in Theorem 5 is unnecessary.

Weight matrices

Using the linear forms (4) and (15), define the problems $\mathcal{L}_k$, $k = 1, 2, 3$, and the Weyl-Yurko matrix $M(\lambda) = [m_{jk}(\lambda)]_{j,k=1}^4$.

$$\mathcal{L}_k : U_s(y) = 0, \quad s = \overline{1, k}, \qquad V_j(y) = 0, \quad j = \overline{1, 4-k}. \tag{18}$$

Simplicity condition

Assume that the eigenvalues of $\mathcal{L}_1$ and $\mathcal{L}_2$ are simple.

Then, all the poles Λ of the Weyl-Yurko matrix elements are simple:

$$M(\lambda) = \frac{M_{\langle -1 \rangle}(\lambda_0)}{\lambda - \lambda_0} + M_{\langle 0 \rangle}(\lambda_0) + M_{\langle 1 \rangle}(\lambda_0)(\lambda - \lambda_0) + \dots, \quad \lambda_0 \in \Lambda. \tag{19}$$

Define the weight matrices: $\mathcal{N}(\lambda_0) := M_{\langle 0 \rangle}^{-1}(\lambda_0) M_{\langle -1 \rangle}(\lambda_0)$, $\lambda_0 \in \Lambda$.

Theorem 6 (Yurko — smooth, Bond. — non-smooth)

Under the simplicity condition, the spectral data $\{\lambda_0, \mathcal{N}(\lambda_0)\}_{\lambda_0 \in \Lambda}$ uniquely specify p, q, a, b, c.

Multiple eigenvalues:

- Buterin, S.A. On inverse spectral problem for non-selfadjoint Sturm-Liouville operator on a finite interval, J. Math. Anal. Appl. (2007).

McLaughlin's problem

Theorem 7 (Bond., 2023)

Under the simplicity condition, the spectral data $\{\lambda_n, \xi_n, \gamma_n\}_{n \geqslant 1}$ *uniquely specify* p, q, a, b, c.

The eigenfunctions fulfill the conditions:

$$U_1(y_n) = U_2(y_n) = 0, \quad V_1(y_n) = V_2(y_n) = 0.$$

Cases:

$$\begin{aligned}
(I)&: \quad U_3(y_n) \neq 0, \quad V_3(y_n) \neq 0 \ \text{(Separation condition)};\\
(II)&: \quad U_3(y_n) \neq 0, \quad V_3(y_n) = 0;\\
(III)&: \quad U_3(y_n) = 0, \quad V_3(y_n) \neq 0;\\
(IV)&: \quad U_3(y_n) = 0, \quad V_3(y_n) = 0.
\end{aligned}$$

Structure of weight matrices

$$(I): \quad \mathcal{N}(\lambda_n) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (II): \quad \mathcal{N}(\lambda_n) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix},$$

$$(III): \quad \mathcal{N}(\lambda_n) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & * & 0 \end{bmatrix}, \quad (IV): \quad \mathcal{N}(\lambda_n) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix},$$

$$\mu_n \in \sigma(\mathcal{L}_1) \setminus \sigma(\mathcal{L}_2): \quad \mathcal{N}(\mu_n) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{bmatrix}.$$

$$\{\lambda_n, \gamma_n, \xi_n\}_{n \geqslant 1} \quad \Rightarrow \quad \{\lambda_0, \mathcal{N}(\lambda_0)\}_{\lambda_0 \in \Lambda}.$$

This idea proves Theorem 7.

Counterexample

In the uniqueness theorem for Barcilon's problem (Theorem 3), the separation condition cannot be omitted.

Idea: Specify the discrete spectral data $\{\lambda_{n,k}, \mathcal{N}_{n,k}\}$ (corresponding to the Weyl-Yurko matrix $M(\lambda)$) and construct $p(x)$, $q(x)$ by using the method of spectral mappings.

Notations:
For $k \in \{1, 2, 3\}$, denote by $\{\lambda_{n,k}\}_{n \geqslant 1}$ the eigenvalues of $\mathcal{L}_k$ and $\mathcal{N}_{n,k} := \mathcal{N}(\lambda_{n,k})$.

$$\begin{aligned}
\mathcal{L}_1 : \quad & y(0) = 0, \quad y(1) = y'(1) = y''(1) = 0, \\
\mathcal{L}_2 : \quad & y(0) = y'(0) = 0, \quad y(1) = y'(1) = 0, \\
\mathcal{L}_3 : \quad & y(0) = y'(0) = y''(0), \quad y(1) = 0.
\end{aligned}$$

$$\mathcal{L}_2 = \mathcal{L}_2^*, \quad \mathcal{L}_1 = \mathcal{L}_3^*.$$

Counterexample

Model problem: $\tilde{p} = \tilde{q} = 0$.

- $\{\tilde{\lambda}_{n,2}\}_{n\geqslant 1}$ are simple and positive.
- $\{\tilde{\lambda}_{n,1}\}_{n\geqslant 1}$ and $\{\tilde{\lambda}_{n,3}\}_{n\geqslant 1}$ are simple and negative.

$$\tilde{\mathcal{N}}_{n,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \tilde{\beta}_{n,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathcal{N}}_{n,1} = \tilde{\mathcal{N}}_{n,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tilde{\beta}_{n,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\beta}_{n,3} & 0 \end{bmatrix},$$

$$\tilde{\lambda}_{n,1} = \tilde{\lambda}_{n,3}, \quad \tilde{\beta}_{n,1} = \tilde{\beta}_{n,3}, \quad n \geqslant 1,$$

$$\lambda_{1,1} = \lambda_{1,2} = \lambda_{1,3} = \tilde{\lambda}_{1,2}, \quad \mathcal{N}_{1,1} = \mathcal{N}_{1,2} = \mathcal{N}_{1,3} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{bmatrix}, \quad \gamma > 0,$$

$$\lambda_{n,k} := \tilde{\lambda}_{n,k}, \quad \mathcal{N}_{n,k} := \tilde{\mathcal{N}}_{n,k}, \quad n \geqslant 2, \quad k = 1, 2, 3,$$

Theorem 8 (Bond., 2024)

For each $\gamma > 0$, there exist unique functions p_γ and q_γ of $C^\infty[0,1]$ such that equation (3) with the coefficients $p = p_\gamma$ and $q = q_\gamma$ has the spectral data $\{\lambda_{n,k}, \mathcal{N}_{n,k}\}_{n\geqslant 1,\, k=1,2,3}$. The corresponding three spectra $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, $\mathfrak{S}_{23}$ do not depend on the parameter $\gamma > 0$.

Conclusions

- Under the separation condition, solution of Barcilon's problem is unique.
- In general, solution of Barcilon's problem can be non-unique.
- Under the separation condition, Barcilon's and McLaughlin's problems are equivalent.
- For McLaughlin's problem, the uniqueness holds without the separation condition.
- Barcilon's and McLaughlin's problems can be interpreted within the framework of the general approach.

Papers

1. Guan, A.-W.; Yang, C.-F.; Bondarenko, N.P. Solving Barcilon's inverse problems for the method of spectral mappings, arXiv:2304.05747.
2. Guan, A.-W.; Yang, C.-F.; Bondarenko, N.P. A class of higher order inverse spectral problems, arXiv:2402.18343, accepted in Acta Mathematica Sinica.
3. Bondarenko, N.P. McLaughlin's inverse problem for the fourth-order differential operator, arXiv:2312.15988.
4. Bondarenko, N.P. Counterexample to Barcilon's uniqueness theorem for the fourth-order inverse spectral problem, Results in Mathematics 79 (2024), Article Number 183.

Thank you for your attention!