Spectral shift functions and Dirichlet-to-Neumann maps

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Contents

PART I

Spectral shift function

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General assumption

$\mathcal{H}$ Hilbert space, A, B selfadjoint (unbounded) operators in $\mathcal{H}$

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I. M. Lifshitz, 1952

$B - A$ **finite rank** operator. Then exists $\xi : \mathbb{R} \to \mathbb{R}$ such that *formally*

$$\operatorname{tr}\left(\varphi(B) - \varphi(A)\right) = \int_{\mathbb{R}} \varphi'(t)\xi(t)\,dt$$

Krein's spectral shift function (1953 and 1962)

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Theorem

Assume $B - A$ **trace class** operator, i.e. $B - A \in \mathfrak{S}_1$.

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Corollary

If $\delta = (a, b)$ and $\overline{\delta} \cap \sigma_{\text{ess}}(A) = \emptyset$ then

$$\xi(b-) - \xi(a+) = \dim \operatorname{ran} E_B(\delta) - \dim \operatorname{ran} E_A(\delta)$$

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If $\delta = (a, b)$ and $\overline{\delta} \cap \sigma_{\mathrm{ess}}(A) = \emptyset$ then

$$\xi(b-) - \xi(a+) = \dim \operatorname{ran} E_B(\delta) - \dim \operatorname{ran} E_A(\delta)$$

- Spectral shift function for U, V unitary, $V - U \in \mathfrak{S}_1$

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$$(B-\lambda)^{-1}-(A-\lambda)^{-1}\in\mathfrak{S}_1,\qquad \lambda\in\rho(A)\cap\rho(B). \tag{1}$$

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Birman-Krein formula

Assume (1). The scattering matrix $\{S(\lambda)\}$ of $\{A,B\}$ satisfies

$$\det S(\lambda)=e^{-2\pi i\xi(\lambda)}\qquad\text{for a.e. }\lambda\in\mathbb{R}$$

Krein's spectral shift function: Generalizations

L.S. Koplienko 1971

Assume $\rho(A) \cap \rho(B) \cap \mathbb{R} \neq \emptyset$ and for some $m \in \mathbb{N}$:

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Some references and contributors

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PART II

Representation of the SSF via Weyl function

Quasi boundary triples

$S \subset S^*$ closed symmetric operator in $\mathcal{H}$ with infinite defect

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Def. [Bruk76, Kochubei75; DerkachMalamud95; B. Langer07]

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Here $(Tf, g) - (f, Tg) = (f|_{\partial\Omega}, \partial_\nu g|_{\partial\Omega}) - (\partial_\nu f|_{\partial\Omega}, g|_{\partial\Omega})$.
Choose $\mathcal{G} = L^2(\partial\Omega)$, $\Gamma_0 f := \partial_\nu f|_{\partial\Omega}$, $\Gamma_1 f := f|_{\partial\Omega}$.

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Remarks

Representation of SSF via M-function:

- Rank 1, $k = 0$ [LangerSnooYavrian'01]
- Rank $n < \infty$, $k = 0$ [B. MalamudNeidhardt'08]
- Other representation via modified perturbation determinant for M for $k = 0$ [MalamudNeidhardt'15]

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- $k = 0$ [B. MalamudNeidhardt'17] [MantilePosilicanoSini'16]

PART III

Applications

Example 1: Robin boundary conditions

$$A_{\beta_0} f = -\Delta f + Vf, \quad \operatorname{dom} A_{\beta_0} = \{ f \in H^2(\Omega) : \beta_0 f|_{\partial\Omega} = \partial_\nu f|_{\partial\Omega} \}$$

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Note: Extends to more general problems [SjöstrandZworski'99]

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Note: ξ continuous for $t \geq 0$, although $N_{A_+} - N_{B_+}$ step function

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$$\sigma_{\mathrm{ess}}(A_{\delta,\alpha}) = [0,\infty) \quad \text{and} \quad \sigma(A_{\delta,\alpha}) \cap (-\infty, 0) \text{ finite}$$

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$$\sigma_{\mathrm{ess}}(A_{\delta,\alpha}) = [0, \infty) \quad \text{and} \quad \sigma(A_{\delta,\alpha}) \cap (-\infty, 0) \;\text{ finite}$$

see [Exner et al. 80s -] [others....] for spectral properties of $A_{\delta,\alpha}$;

Example 3: δ-potentials on hypersurfaces

- Ω_+ bdd. domain in $\mathbb{R}^n$, $\mathcal{C} := \partial\Omega_+$ smooth, $\Omega_- := \overline{\Omega}_+^c$
- $V \in L^\infty(\mathbb{R}^n)$ real and $\alpha, \alpha^{-1} \in L^\infty(\mathcal{C})$ real

Consider the pair $\{A_{\delta,\alpha}, A\}$ with 'free' operator

$$Af = -\Delta f + Vf, \qquad \operatorname{dom} A = H^2(\mathbb{R}^n),$$

and the operator $-\Delta + V + \alpha\delta_{\mathcal{C}}$ given by

$$A_{\delta,\alpha} f = -\Delta f + Vf,$$

$$\operatorname{dom} A_{\delta,\alpha} = \left\{ f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in H^{3/2}_\Delta(\mathbb{R}^n \setminus \mathcal{C}) : \begin{array}{c} f_+|_{\mathcal{C}} = f_-|_{\mathcal{C}}, \\ \alpha f_\pm|_{\mathcal{C}} = \partial_{\nu_+} f_+|_{\mathcal{C}} + \partial_{\nu_-} f_-|_{\mathcal{C}} \end{array} \right\}$$

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Example 3: δ-potentials on hypersurfaces

Let

$$\mathcal{E}(\lambda) = \big(\mathcal{D}_+(\lambda) + \mathcal{D}_-(\lambda)\big)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

Dirichlet-to-Neumann maps $\mathcal{D}_\pm(\lambda) f_{\pm,\lambda}|_\mathcal{C} = \partial_{\nu_\pm} f_{\pm,\lambda}|_\mathcal{C}$ in $L^2(\mathcal{C})$

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Theorem: For $k \geq \frac{n-3}{4}$ one has

- $(A_{\delta,\alpha} - \lambda)^{-(2k+1)} - (A - \lambda)^{-(2k+1)} \in \mathfrak{S}_1\big(L^2(\mathbb{R}^n)\big)$

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- $(A_{\delta,\alpha} - \lambda)^{-(2k+1)} - (A - \lambda)^{-(2k+1)} \in \mathfrak{S}_1\big(L^2(\mathbb{R}^n)\big)$
- Spectral shift function for $\{A_{\delta,\alpha}, A\}$:

$$\xi(t) = \sum_j \lim_{\varepsilon \to +0} \frac{1}{\pi}\Big(\mathrm{Im}\big(\log(\mathcal{M}_0(t+i\varepsilon)) - \log(\mathcal{M}_\alpha(t+i\varepsilon))\big)\varphi_j, \varphi_j\Big),$$

where $\mathcal{M}_\alpha(\lambda) = \frac{1}{c-\alpha}\big(\alpha\mathcal{E}(\lambda) - I_{L^2(\mathcal{C})}\big)\big(c\mathcal{E}(\lambda) - I_{L^2(\mathcal{C})}\big)^{-1}$, and $c > 0$ such that $\alpha(x) < c$ for all $x \in \mathcal{C}$.

Example 3: δ-potentials on hypersurfaces

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Theorem: For $n = 2, 3$ one has

- $(A_{\delta,\alpha} - \lambda)^{-1} - (A - \lambda)^{-1} \in \mathfrak{S}_1\big(L^2(\mathbb{R}^n)\big)$
- Spectral shift function for $\{A_{\delta,\alpha}, A\}$:

$$\xi(t) = \lim_{\varepsilon \to +0} \frac{1}{\pi}\Big(\mathrm{Im}\big(\log(\mathcal{M}_0(t+i\varepsilon)) - \log(\mathcal{M}_\alpha(t+i\varepsilon))\big)\Big),$$

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References

This talk was based on the papers

- JB, F. Gesztesy, S. Nakamura
  Spectral shift functions and Dirichlet-to-Neumann maps
  Math. Ann. 371 (2018), 1255-1300
- JB, F. Gesztesy, S. Nakamura
  A spectral shift function for Schrödinger operators with singular interactions
  Operator Theory Adv. Appl. 268 (2018), 89-110

Thank you for your attention