Soliton solutions to a system of nonlinear evolution equations associated with a third-order ordinary linear differential equation

Tuncay Aktosun

Department of Mathematics
University of Texas at Arlington
Arlington, TX 76019, USA

August 12, 2024
AMP 2024 Online Conference

(joint with A. Choque-Rivero, I. Toledo, and M. Unlu)

Outline

1. Inverse Scattering Transform method
2. Integrable system of nonlinear equations
3. Sawada–Kotera equation, Kaup–Kupershmidt equation, bad Boussinesq equation
4. Direct and inverse scattering for the third-order equation
5. Riemann–Hilbert formulation of the inverse scattering problem
6. Relevant prior work
7. Reflectionless case, explicit solutions
8. Mathematica demonstration

Inverse Scattering Transform method

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{direct scattering for LODE at } t=0} & S(\lambda,0) \\
\text{solution to NPDE} \downarrow & & \downarrow \text{time evolution of scattering data} \\
u(x,t) & \xleftarrow[\text{inverse scattering for LODE at time } t]{} & S(\lambda,t)
\end{array}
$$

Lax pair (L, A) : L appears in $L\psi = \lambda\psi$ and A determines the time evolution

- the spectral parameter λ does not change in time, i. e. $\lambda_t = 0$.
- the quantity $\psi_t - A\psi$ remains a solution to $L\psi = \lambda\psi$, i.e. $L(\psi_t - A\psi) = \lambda(\psi_t - A\psi)$.
- the quantity $L_t + LA - AL$ is the zero multiplication operator.

Integrable system of nonlinear equations

$$\begin{cases} Q_t + Q_{xxxxx} + 5\,Q\,Q_{xxx} + 5\,Q_x\,Q_{xx} + 5\,Q^2\,Q_x + 15\,P\,Q_{xx} - 30\,P\,P_x + 15\,P_x\,Q_x = 0, \\ P_t + P_{xxxxx} + 5\,Q\,P_{xxx} + 15\,Q_x\,P_{xx} + 20\,P_x\,Q_{xx} + 5\,Q^2\,P_x + 10\,P\,Q_{xxx} \\ \qquad\qquad - 15\,P\,P_{xx} + 10\,P\,Q\,Q_x - 15\,(P_x)^2 = 0. \end{cases}$$

- obtained by using $L_t + LA - AL = 0$ with the Lax pair (L, A)

  $L = D^3 + QD + P,$

  $A = 9D^5 + 15QD^3 + (15P + 15Q_x)D^2 + (10Q_{xx} + 15P_x + 5Q^2)D + (10P_{xx} + 10QP).$
- analyze the inverse scattering problem for

  $\psi''' + Q(x,t)\,\psi' + P(x,t)\,\psi = k^3\,\psi, \quad x \in \mathbb{R}, \ t \geq 0$
- relevant wavefunctions, time-evolved wavefunctions
- scattering coefficients, time-evolved scattering coefficients
- bound-state information, time-evolved bound-state information

Special cases of the integrable nonlinear system

- Sawada–Kotera equation when $P(x,t) \equiv 0$ and $Q(x,t)$ real valued

  $$Q_t + Q_{xxxxx} + 5Q_x Q_{xx} + 5QQ_{xxx} + 5Q^2 Q_x = 0, \quad x \in \mathbb{R}$$

- Kaup–Kupershmidt equation when $P(x,t) \equiv Q_x(x,t)$ and $Q(x,t)$ real valued

  $$Q_t + Q_{xxxxx} + 5Q_x Q_{xx} + 5QQ_{xxx} + 5Q^2 Q_x = 0, \quad x \in \mathbb{R}$$

A related integrable system

- bad Boussinesq system $\begin{cases} q_t = -3p_x, \\ p_t = -q_{xxx} + 8qq_x. \end{cases}$
- obtained by using $L_t + LA - AL = 0$ with the Lax pair (L, A)

  $L = iD^3 - 2iqD + (p - iq'), \quad A = 3iD^2 - 4iq,$
- selfadjoint linear operator L
- bad Boussinesq equation $q_{tt} - 3q_{xxxx} + 12(q^2)_{xx} = 0, \quad x \in \mathbb{R}$
- analyze the inverse scattering problem for

  $\psi''' - 2q(x)\,\psi' - [q'(x) + i\,p(x)]\,\psi = k^3\,\psi, \quad x \in \mathbb{R}$
- scattering coefficients, time-evolved scattering coefficients
- bound-state information, time-evolved bound-state information
- explicit closed-form solutions in the reflectionless case

Direct scattering problem for the third-order equation

$$\psi''' + Q(x)\,\psi' + P(x)\,\psi = k^3\,\psi, \quad x \in \mathbb{R}$$

- Jost solutions $f(k,x)$ and $g(k,x)$
- two other fundamental solutions $h^{\text{up}}(k,x)$ and $h^{\text{down}}(k,x)$
- scattering coefficients $T_{\text{l}}(k)$, $L(k)$, $M(k)$, $T_{\text{r}}(k)$, $R(k)$, $N(k)$
- bound states at $k = k_j$ with dependency constant D_j for $1 \le j \le \mathbf{N}$
- bound-state information, time-evolved bound-state information
- time evolution of fundamental solutions, scattering coefficients
- time evolution of bound-state dependency constants
- explicit closed-form solutions in the reflectionless case

Scattering solutions

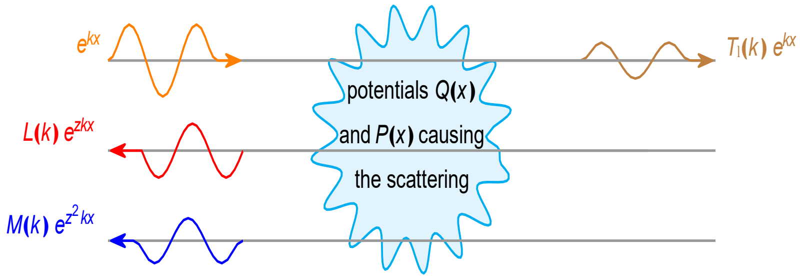
$$\psi''' + Q(x)\,\psi' + P(x)\,\psi = k^3\,\psi, \quad x \in \mathbb{R}$$

- scattering as a result of $Q(x)$ and $P(x)$ vanishing as $x \to \pm\infty$
- unperturbed problem $\psi''' = k^3\,\psi, \quad x \in \mathbb{R}$
- asymptotic behavior of any solution

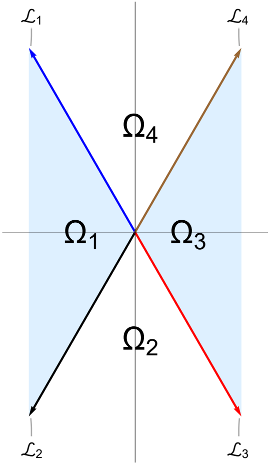
$$\psi(k,x) \sim \begin{cases} a_1(k)\,e^{kx} + a_2(k)\,e^{zkx} + a_3(k)\,e^{z^2 kx}, & x \to +\infty, \\ a_4(k)\,e^{kx} + a_5(k)\,e^{zkx} + a_6(k)\,e^{z^2 kx}, & x \to -\infty \end{cases}$$

with $z := e^{2\pi i/3}$, satisfying $z^3 - 1 = (z-1)(z^2+z+1) = 0.$

Scattering scenario



Domains in the complex k-plane



$\overline{\Omega}_1$	k-domain of $f(k,x)$
$\overline{\Omega}_2$	k-domain of $h^{\text{down}}(k,x)$
$\overline{\Omega}_3$	k-domain of $g(k,x)$
$\overline{\Omega}_4$	k-domain of $h^{\text{up}}(k,x)$
$\mathcal{L}_1$	domain of $L(k)$ and $T_{\text{l}}(k)$
$\mathcal{L}_2$	domain of $M(k)$ and $T_{\text{l}}(k)$
$\mathcal{L}_3$	domain of $R(k)$ and $T_{\text{r}}(k)$
$\mathcal{L}_4$	domain of $N(k)$ and $T_{\text{r}}(k)$

Jost solutions and scattering coefficients

- left Jost solution $f(k,x)$ satisfying $\lim_{x\to+\infty} f(k,x) \sim e^{kx}, \quad k \in \overline{\Omega}_1$
  - $$\lim_{x\to-\infty} f(k,x) \sim \begin{cases} T_{\rm l}(k)^{-1}\, e^{kx} + L(k)\, T_{\rm l}(k)^{-1}\, e^{zkx}, & k \in \mathcal{L}_1, \\ T_{\rm l}(k)^{-1}\, e^{kx}, & k \in \Omega_1, \\ T_{\rm l}(k)^{-1}\, e^{kx} + M(k)\, T_{\rm l}(k)^{-1}\, e^{z^2kx}, & k \in \mathcal{L}_2 \end{cases}$$

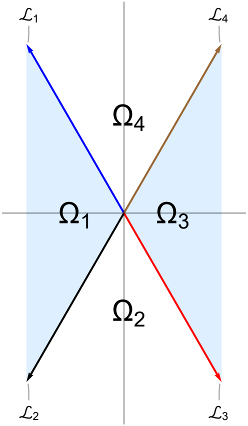
  $T_{\rm l}(k)$ left transmission coeff

  $L(k),\ M(k)$ left primary and left secondary reflection coeffs

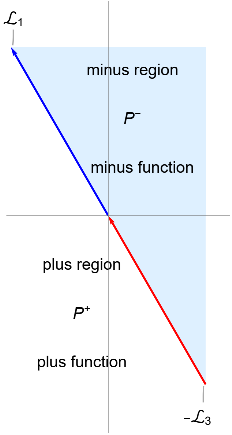
- right Jost solution $g(k,x)$ satisfying $\lim_{x\to-\infty} f(k,x) \sim e^{kx}, \quad k \in \overline{\Omega}_3$
  - $$\lim_{x\to+\infty} g(k,x) \sim \begin{cases} T_{\rm r}(k)^{-1}\, e^{kx} + R(k)\, T_{\rm r}(k)^{-1}\, e^{zkx}, & k \in \mathcal{L}_3, \\ T_{\rm r}(k)^{-1}\, e^{kx}, & k \in \Omega_3, \\ T_{\rm r}(k)^{-1}\, e^{kx} + N(k)\, T_{\rm r}(k)^{-1}\, e^{z^2kx}, & k \in \mathcal{L}_4 \end{cases}$$

  $T_{\rm r}(k)$ right transmission coeff

  $R(k),\ N(k)$ right primary and right secondary reflection coeffs

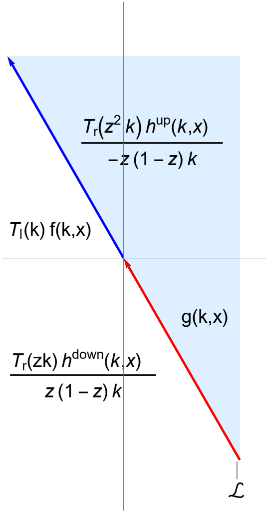


$\mathcal{L}_1$
$\mathcal{L}_4$
Ω_4
Ω_1
Ω_3
Ω_2
$\mathcal{L}_2$
$\mathcal{L}_3$



$\mathcal{L}_1$
minus region
P^-
minus function
plus region
P^+
plus function
$-\mathcal{L}_3$

Riemann–Hilbert problem formulation



Choose $M(k) \equiv 0,\; N(k) \equiv 0.$

$$\Phi_+(k,x) := \begin{cases} T_l(k)\, f(k,x), & k \in \overline{\Omega}_1, \\ \dfrac{T_r(zk)\, h^{\text{down}}(k,x)}{z(1-z)k}, & k \in \overline{\Omega}_2 \end{cases}$$

$$\Phi_-(k,x) := \begin{cases} g(k,x), & k \in \overline{\Omega}_3, \\ \dfrac{T_r(z^2k)\, h^{\text{up}}(k,x)}{-z(1-z)k}, & k \in \overline{\Omega}_4 \end{cases}$$

Riemann–Hilbert problem formulation

$$\Phi_+(k,x) = \Phi_-(k,x) + J(k,x), \qquad k \in \mathcal{L}$$

jump on $\mathcal{L}$ $\qquad J(k,x) = \begin{cases} L(k)\, T_{\rm l}(zk)\, f(zk,x), & k \in \mathcal{L}_1, \\ -R(k)\, \dfrac{T_{\rm r}(zk)}{T_{\rm r}(k)}\, g(zk,x), & k \in -\mathcal{L}_3 \end{cases}$

- solve the Riemann–Hilbert problem and obtain $\Phi_+(k,x)$
- recover $f(k,x)$ from $\Phi_+(k,x)$
- recover $Q(x)$ and $P(x)$ from $f(k,x)$ as $k \to \infty$ in $k \in \overline{\Omega}_1$

$$f(k,x) = e^{kx}\left[1 + \frac{u_1(x)}{k} + \frac{u_2(x)}{k^2} + O\left(\frac{1}{k^3}\right)\right],$$

$$Q(x) = -3\,\frac{du_1(x)}{dx}, \qquad x \in \mathbb{R},$$

$$P(x) = 3\left[u_1(x)\,\frac{du_1(x)}{dx} - \frac{d^2u_1(x)}{dx^2} - \frac{du_2(x)}{dx}\right], \qquad x \in \mathbb{R}.$$

Relevant prior work

- Kaup (1980)
  - Started the study of direct and inverse scattering problems for the third-order equation
  - Unsuccessfully sought a Marchenko-like integral equation
- Beals, Coifman (1984,1987)
  - Direct and inverse scattering for *n*th order equations
  - Riemann–Hilbert formulation
  - no Marchenko-like integral equation
- Deift, Tomei, Trubowitz (1982)
  - direct and inverse scattering related to the bad Boussinesq equation
  - assumptions $T_{\mathrm{l}}(k) \equiv 1$, $T_{\mathrm{r}}(k) \equiv 1$, $M(k) \equiv 0$, $N(k) \equiv 0$
  - Riemann-Hilbert problem, selfadjoint differential operator, analytic continuations
  - Marchenko-like integral equation, no bound states, no soliton-like solutions
- Hirota (1989)
  - Hirota's method for particular soliton solutions,
  - Marchenko-like equation for particular soliton solutions to Sawada–Kotera equation
  - no relation to scattering
- Parker (2001)
  - dressing method of Shabat–Zakharov to derive Hirota's integral equation
  - no relation to scattering

Marchenko method for the third-order equation

- modify Riemann–Hilbert problem $\Phi_+(k,x) = \Phi_-(k,x) + J(k,x), \qquad k \in \mathcal{L}$

$$e^{-kx}[\Phi_+(k,x) - 1] = e^{-kx}[\Phi_-(k,x) - 1] + e^{-kx} J(k,x), \qquad k \in \mathcal{L}$$

- apply the Fourier transform along $\mathcal{L}$, parametrized as $k = zs$ with $s \in (-\infty, +\infty)$

$$K(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds\, e^{isy} \left[e^{-zsx} [\Phi_+(zs,x) - 1] \right], \qquad k \in \mathcal{L}$$

Reflectionless case, soliton-like solutions

- bound-state information consisting of bound-state energies and dependency constant

  transmission coefficients $T_{\mathrm{l}}(k) = \prod_{j=1}^{\mathbf{N}} \frac{(k+k_j)(k+k_j^*)}{(k-k_j)(k-k_j^*)}, \quad T_{\mathrm{r}}(k) = \frac{1}{T_{\mathrm{l}}(k)}$

  $k_j = iz\eta_j$ and $k_j^* = -iz^2\eta_j$ for $1 \le j \le \mathbf{N}$ with $z = e^{2\pi i/3}$ and $0 < \eta_1 < \cdots < \eta_{\mathbf{N}}$

  dependency constants $\quad f(k_j, x) = D_j(t)\, g(zk_j, x)$

  dependency constants $\quad D_j(t) = E_j\, e^{-9\sqrt{3}\eta_j^5 t}$

- use the bound-state information $\{k_j, E_j\}_{j=1}^{\mathbf{N}}$ as input to the Riemann-Hilbert problem

$$e^{-kx}\,\Phi_+(k,x) = e^{-kx}\,\Phi_-(k,x), \qquad k \in \mathcal{L}$$

- apply restrictions on dependency constants if $P(x,t) \equiv 0$ and $Q(x,t)$ real
- apply restrictions on dependency constants if $P(x,t) \equiv Q_x(x,t)$ and $Q(x,t)$ real
- explicit construction of $Q(x,t)$, $P(x,t)$, $f(k,x)$, $g(k,x)$ and all relevant quantities

References

R. Beals and R. Coifman, *Scattering and inverse scattering for first order systems*, Comm. Pure Appl. Math. **37**, 39–90 (1984).

R. Beals and R. Coifman, *Scattering and inverse scattering for first-order systems: II*, Inverse Probl. **3** 577–593 (1987).

P. Deift, C. Tomei, and E. Trubowitz, *Inverse scattering and the Boussinesq equation*, Comm. Pure Appl. Math. **35**, 567–628 (1982).

R. Hirota, *Soliton solutions to the BKP equations. II. The integral equation*, J. Phys. Soc. Jpn. **58**, 2705–2712 (1989).

D. Kaup, *On the inverse scattering problem for cubic eigenvalue problems of the class* $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, Stud. Appl. Math. **62**, 189–216 (1980).

A. Parker, *A reformulation of the 'dressing method' for the Sawada–Kotera equation*, Inverse Probl. **17**, 885–895 (2001).

K. Sawada and T. Kotera, *A method for finding N-soliton solutions of the KdV equation and KdV-like equation*, Prog. Theor. Phys. **51**, 1355–1367 (1974).